Ph. D. Thesis Applied Mathematics

Oksana Shatalov

ISOMETRIC EMBEDDINGS $\ell_2^m \to \ell_p^n$ AND CUBATURE FORMULAS OVER CLASSICAL FIELDS

Supervisor: Prof. Y.I. Lyubich

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I AM SINCERELY GRATEFUL TO MY SCIENTIFIC SUPERVISOR,
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Contents

Abstract

The subject of the present work arose in a connection with well known problem about almost Euclidean subspaces of normed spaces. The Euclidean subspaces in the real space ℓ_p^n are just the images of isometric embeddings $\ell_2^m \to \ell_p^n$.

The problem of isometric embeddings of real spaces $\ell_2^m \to \ell_p^n$ was considered by Lyubich-Vaserstein and independently by Reznick where an equivalence between isometric embeddings of real spaces $\ell_2^m \to \ell_p^n$ and multi-dimensional cubature formulas on the unit sphere was established. In the complex case some results were obtained by König.

In the present work the isometric embeddings $\ell_2^m \to \ell_p^n$ with even p (a necessary condition) over any classical field $\mathbf{K}$, i.e. over real field $\mathbf{R}$ or complex field $\mathbf{C}$ or quaternionic field $\mathbf{H}$, are investigated. The quaternionic isometric embeddings $\ell_2^m \to \ell_p^n$ are considered here for the first time. A unified theory where the differences between concrete fields are reduced as soon as possible is developed. An essential ingredient of this theory is a generalization of the equivalence between isometric embeddings and cubature formulas to any field $\mathbf{K}$.

For given m, p the existence of isometric embedding $\ell_2^m \to \ell_p^n$ with large n follows from Hilbert's identity and a geometrical argument. This also yields an upper bound for the minimal number $n = N_{\mathbf{K}}(m, p)$ such that an isometric embedding $\ell_2^m \to \ell_p^n$ does exist.

An inductive (on m) construction of isometric embeddings is developed. In some cases the latter allows us to improve the upper bounds for $N_{\mathbf{K}}(m, p)$. This approach also yields some new isometric embeddings over $\mathbf{K}$ using some known ones over $\mathbf{R}$. In particular, it turns out that $N_{\mathbf{H}}(7, 10) \leq 6486480$.

A lower bound for $N_{\mathbf{K}}(m, p)$ is obtained using the cubature formula theory. A criterion of exactness (tightness) of the lower bound is established and applied to some concrete situations. Any tight cubature formula has equal weights, in other words, their supports are designs.

A group orbit method to construct real isometric embeddings is extended to the complex and quaternionic situations. The finite subgroups of $SU(2)$ are systematically used to obtain the isometric embeddings $\ell_2^2 \to \ell_p^n$ over $\mathbf{C}$. In particular, some complex isometric embeddings $\ell_2^2 \to \ell_8^{10}$ and $\ell_2^2 \to \ell_{10}^{12}$ are obtained. The first of them is minimal. The second one yields that a tight complex 5-design does exist. This is the affirmative answer to the Bannai-Hoggar question, so Hoggar's later statement about non-existence of such a design turns out to be disproved.

For the reader convenience a wide algebraic and analytic background is included, in particular, there is Witt's Theorem on unitary equivalence of vector systems and the Addition Theorem for spherical harmonics and its generalization to the complex and quaternionic spheres.

List of Symbols[1]

Chapter 1

$\mathbf{R}$	–	the set of real numbers
$\mathbf{C}$	–	the set of complex numbers
$\mathbf{N}$	=	$\{0, 1, 2, \ldots\}$, the set of natural numbers
$\mathbf{H}$	–	the set of quaternions
$\mathbf{i}$	–	standard imagine unit in $\mathbf{C}$
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	–	standard imagine units in $\mathbf{H}$
$\mathbf{K}$	–	basis field
$\mathrm{Span}(u_1, \ldots, u_n)$	–	linear span of vectors $u_1, \ldots, u_n$
$\dim F$	–	dimension of space F over $\mathbf{K}$ (if $\mathbf{K}$ is fixed a priori)
$M_{n,m}(\mathbf{K})$	–	space of all $n \times m$ matrices over $\mathbf{K}$
$M_{n,n}(\mathbf{K})$	=	$M_{n,n}(\mathbf{K})$
$\mathbf{K}^n$	–	set of columns of height n with scalar entries
$(\mathbf{K}^n)'$	–	set of rows of length n with scalar entries
$[\mathbf{K} : \mathbf{L}]$	–	degree of finite extension $\mathbf{K}$ over $\mathbf{L}$
$\dim_{\mathbf{L}} F$	–	dimension of space F over $\mathbf{L}$
$[\tau_{ik}]$	–	matrix with elements τ_{ik}
$\dot{+}$	–	direct sum
$\mathrm{codim} X$	–	codimension of subspace X
$\mathrm{GL}_n(\mathbf{K})$	–	set of all invertible $n \times n$ matrices
$T_n(\mathbf{K})$	–	set of transition $n \times n$ matrices
e	–	unit matrix
$[\xi]'$	–	transpose of $[\xi]$
t^{-1}	–	inverse matrix of t
F'	–	set of all linear functionals on F
δ_{ik}	=	$\begin{cases} 1 & (i = k \\ 0 & (i \neq k) \end{cases}$, Kronecker's delta.
$\mathrm{Hom}(E, F)$	–	set of all homomorphisms from E into F
$\mathrm{End}(E)$	–	set of all endomorphisms of E
$L(E)$	–	set of all linear operators in E
Im	–	image
Ker	–	kernel
rank	–	rank
def	–	defect
id	–	identity operator
$L^{\#}(E)$	–	set of all invertible linear operators in E
$\mathrm{Aut}(E)$	–	set of all automorphisms in E

[1]All symbols are listed in order of their appearance in Chapters 1 – 4.

$E \approx F$	$-$	E isomorphic to F
$\otimes$	$-$	tensor product
$\mathrm{mat}(f)$	$-$	matrix of homomorphism f
f'	$-$	adjoint mapping
$\langle \cdot, \cdot \rangle$	$-$	inner product in Euclidean space
$\mathbf{k}$	$=$	$\{\lambda : \ \lambda \in \mathbf{K}, \ \overline{\lambda} = \lambda\}$
$\Re\mathrm{e}(\alpha)$	-	real part of α
$\Im\mathrm{m}(\alpha)$	-	imagine part of α
$\perp$	$-$	orthogonality relation
$\mathrm{pr}_X v$	$-$	orthogonal projection of vector v on space X
deg	-	degree
$\oplus$	$-$	orthogonal sum
$X^\perp$	$-$	direct complement of subspace X
E^*	$-$	conjugate space to E
$U(E)$	$-$	set of unitary operators in E
$O(E)$	$-$	orthogonal group
$\mathbf{Q}$	$-$	the set of rational numbers
$U_m(\mathbf{K})$	$-$	set of unitary $m \times m$ matrices
t^*	$-$	conjugate matrix of t
$\Vert \cdot \Vert$	$-$	norm
$B(E)$	$-$	closed unit ball in E
$S(E)$	$-$	unit sphere in E
$\widehat{z}$	$-$	$\dfrac{z}{\Vert z \Vert}$, z is a vector
$\Vert \cdot \Vert$	$-$	ℓ_p-norm
$\ell^n_{p,\mathbf{K}}$	$=$	$(\mathbf{K}^n, \Vert \cdot \Vert_p), \quad 1 \le p \le \infty$
ℓ^n_p	$\equiv$	$\ell^n_{p,\mathbf{K}}$ if $\mathbf{K}$ is fixed a priori
$\mathrm{Iso}(E)$	$-$	set of all surjective isometries of E
$\mathrm{Arg}\beta$	$-$	argument of $\beta \in \mathbf{C}$
$E_\mathbf{R}$	-	the realification of an Euclidean space E
δ	$=$	$[\mathbf{K} : \mathbf{R}]$, degree of $\mathbf{K}$ over $\mathbf{R}, \ \mathbf{C}, \ \mathbf{H}$
$\mathbf{S}^{(m-1)}$	$-$	$(m-1)$-sphere in $\mathbf{R}^m$

`Chapter 2`

$P_k^{\alpha,\beta}$	$-$	Jacobi polynomial
$\omega_{\alpha,\beta}(u)$	$=$	$(1-u)^\alpha (1+u)^\beta$, Jacobi weight
$\tau_{\alpha,\beta}$	$-$	$\int_{-1}^1 \omega_{\alpha,\beta}(u)\, du$
$\Omega_{\alpha,\beta}$	$-$	$\omega_{\alpha,\beta}/\tau_{\alpha,\beta}$, normalaized Jacobi weight
ω_q	$=$	$\omega_{\frac{q-3}{2},\frac{q-3}{2}}$
τ_q	$=$	$\tau_{\frac{q-3}{2},\frac{q-3}{2}}$
Ω_q	$=$	$\Omega_{\frac{q-3}{2},\frac{q-3}{2}}$

C_k^ν	–	Gegenbauer polynomial
$\binom{m}{n}$	–	binomial coefficient
$n!!$	=	$\begin{cases} 1\cdot 3\cdot\ldots(n-2)\cdot n & \text{if } n \text{ is odd}, \quad n\ge 1 \\ 2\cdot 4\cdot\ldots(n-2)\cdot n & \text{if } n \text{ is even}, \quad n\ge 2 \\ 1 & \text{if} \quad n=0,-1 \end{cases}$
$[t]$	–	integer part of $t\in\mathbf{R}$
$\Gamma(\cdot)$	–	Gamma function
ε_d	=	$\text{res}(d)(\text{mod}2), \quad t\in\mathbf{R}$
σ	–	normalized Lebesgue measure
$\widetilde{\sigma}_{q-1}$	–	measure (area) induced on the sphere $\mathbf{S}^{q-1}\subset\mathbf{R}^q$
$\Upsilon_{\mathbf{K}}(m,t)$	=	$\left(\int \lvert\langle x,y\rangle\rvert^{2t}\,d\sigma(y)\right)^{-1}, \quad x\in\mathbf{S}(E)$

Chapter 3

Δ	–	Laplacian
$\mathcal{P}(E)$	–	space of all polynomial in E
$\mathcal{H}(E)$	–	space of all harmonic polynomial in E
$\mathtt{Pol}(E)$	–	space of all polynomial functions on $\mathbf{S}(E)$
$\mathtt{Harm}(E)$	–	space of all harmonic polynomial functions on $\mathbf{S}(E)$
$\mathcal{P}_d(E)$	=	$\{\psi\in\mathcal{P}(E) : \deg\psi\le d \text{ or } \psi=0\}$
$\mathcal{H}_d(E)$	–	subspace of harmonic polynomial of $\mathcal{P}_d(E)$
$\mathtt{Pol}_d(E)$	–	space of all polynomial functions of degree $\le d$
$\mathtt{Harm}_d(E)$	–	space of all harmonic polynomial functions of degree $\le d$
$\mathcal{P}(E;d)$	–	space of all forms of degree d
$\mathcal{H}(E;d)$	–	space of all harmonic forms of degree d
$\mathtt{Pol}(E;d)$	–	space of functions from $\mathcal{P}(E;d)$ restricted to $\mathbf{S}(E)$
$\mathtt{Harm}(E;d)$	–	space of spherical harmonics of degree d
$\mathcal{E}_d$	=	$\{k : \ 0\le k\le d, \ k\equiv d(\text{mod } 2)\}$
r	-	restriction
$h_{m,k}$	–	dimension of $\mathtt{Harm}(E;k)$
Π	–	linear space of all polynomials of one variable
Π_d	=	$\{Q\in\Pi : \deg Q\le d\}$
$\Pi^{(0)}$	–	subspace of even polynomials of Π
$\Pi^{(1)}$	–	subspace of odd polynomials of Π
$c_{m,k}(F)$	–	Fourier coefficient of F
$\mathbf{KP}^{m-1}$	–	arithmetic projective space over $\mathbf{K}$
$\mathbf{Z}_2$	–	$\{-e,+e\}$
$U(\mathbf{K})$	–	group of units in $\mathbf{K}$
$\mathcal{P}_{\mathbf{K}}(E)$	–	space of all $U(\mathbf{K})$-invariant polynomial in E
$\mathcal{H}_{\mathbf{K}}(E)$	–	space of all $U(\mathbf{K})$-invariant harmonic polynomial in E
$\mathtt{Pol}_{\mathbf{K}}(E)$	–	space of all $U(\mathbf{K})$-invariant polynomial functions on $\mathbf{S}(E)$

$\mathtt{Harm}_{\mathbf{K}}(E)$	–	space of all $U(\mathbf{K})$-invariant harmonic polynomial functions on $\mathbf{S}(E)$
$\mathcal{P}_{\mathbf{K};d}(E)$	=	$\{\psi \in \mathcal{P}_{\mathbf{K}}(E) : \deg\psi \leq d \text{ or } \psi = 0\}$
$\mathcal{H}_{\mathbf{K};d}(E)$	–	subspace of $U(\mathbf{K})$-invariant harmonic polynomial of $\mathcal{P}_{\mathbf{K};d}(E)$
$\mathtt{Pol}_{\mathbf{K};d}(E)$	–	space of all $U(\mathbf{K})$-invariant polynomial functions of degree $\leq d$
$\mathtt{Harm}_{\mathbf{K};d}(E)$	–	space of all $U(\mathbf{K})$-invariant harmonic polynomial functions of degree $\leq d$
$\mathcal{P}_{\mathbf{K}}(E;d)$	–	space of all $U(\mathbf{K})$-invariant forms of degree d
$\mathcal{H}_{\mathbf{K}}(E;d)$	–	space of all $U(\mathbf{K})$-invariant harmonic forms of degree d
$\mathtt{Pol}_{\mathbf{K}}(E;d)$	–	space of $U(\mathbf{K})$-invariant functions from $\mathcal{P}(E;d)$ restricted to $\mathbf{S}(E)$
$\mathtt{Harm}_{\mathbf{K}}(E;d)$	–	space of $U(\mathbf{K})$-invariant spherical harmonics of degree d
$\hbar_{m,2k}$	–	dimension of $\mathtt{Harm}_{\mathbf{K}}(E;2k)$
$c^{\delta}_{m,k}(f)$	–	Fourier coefficient of f

`Chapter 4`

$A(X)$	–	angle set of spherical code X
supp	–	support
$a(X)$	–	angle set of projective code X
$N_{\mathbf{K}}(m,2t)$	–	minimal number of nodes of projective cubature formula of index $2t$ on $\mathbf{S}(\mathbf{K}^m)$
$\underset{\mathcal{G}}{\mathrm{Ave}}$	–	averaging over group G
$\mathtt{Harm}_G(E;d)$	–	the space of G-invariant spherical harmonics of degree d
$\mathcal{H}_G(E;d)$	–	space of G-invariant harmonic forms of degree d
$\mathtt{Harm}_{\mathbf{K};G}(E;2k)$	–	space of $U(\mathbf{K})$-invariant spherical harmonics of degree d which are also G-invariant
$\mathcal{H}_{\mathbf{K};G}(E;d)$	–	space of $U(\mathbf{K})$-invariant harmonic forms of degree d which are also G-invariant
det	–	determinant
G^+	–	subset of group G consisting of all distinct representative of $G/\mathbf{Z}_2$
$\mathcal{D}_n$	–	binary dihedral group of order $4n$
$\mathcal{T}$	–	binary tetrahedral group
$\mathcal{I}$	–	binary icosahedral group
$\mathcal{C}_5$	–	cyclic group of order 5
ϵ	=	$\exp\left(\frac{\pi i}{4}\right)$

Introduction

The subject of the present work arose in a connection with well known and deeply developed problem about almost Euclidean subspaces of normed spaces. (See [10], [18], [19], [22], [23], [34], [38], [56]. This is far from being a complete list of the publications about the subject.) As a rule, a normed space does not contain Euclidean subspaces of dimensions greater than one. However, the famous Dvoretzky theorem [18] states the existence of almost Euclidean subspaces of all normed spaces of sufficiently big dimensions. Note that in presence of an Euclidean subspace of a dimension $m \geq 2$, an Euclidean plane (a 2-dimensional Euclidean subspace) does exist as well. In the latter case the unit sphere of the given space contains a circle. The spheres of such a kind arise in a natural way rarely. For example, one can prove that the real space ℓ_p^n (the n-dimensional real space provided with ℓ_p-norm) may contain an Euclidean subspace in the only case of even p while there is an example of 2-dimensional Euclidean plane in the real space ℓ_4^3 [33]. An example of Euclidean plane in the real space ℓ_4^{12} was presented in [37]. A proof of existence of m-dimensional Euclidean subspace of the real space ℓ_p^n for sufficiently big n depending on m and p, $n \geq N_{\mathbf{R}}(m,p)$, was outlined in the same work [37]. Such an approach also yields an upper bound for $N_{\mathbf{R}}(m,p)$. Note that the Euclidean subspaces in ℓ_p^n are just the images of isometric embeddings $\ell_2^m \to \ell_p^n$. Later on we prefer to speak about the embeddings.

EXAMPLE 1 (see [33]). The identity

$$\xi_1^4 + \left(\frac{\xi_1 + \xi_2\sqrt{3}}{2}\right)^4 + \left(\frac{\xi_1 - \xi_2\sqrt{3}}{2}\right)^4 = (\xi_1^2 + \xi_2^2)^2 \tag{1}$$

shows that mapping

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \mapsto \begin{bmatrix} \xi_1 \\ (\xi_1 + \xi_2\sqrt{3})/2 \\ (\xi_1 - \xi_2\sqrt{3})/2 \end{bmatrix} \tag{2}$$

is an isometric embedding $\ell_2^2 \to \ell_4^3$. □

EXAMPLE 2 (see [37]). The Lucas identity

$$\sum_{1 \leq i < k \leq 4} (\xi_i + \xi_k)^4 + \sum_{1 \leq i < k \leq 4} (\xi_i - \xi_k)^4 = 6(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2)^2 \tag{3}$$

defines an embedding $\ell_2^4 \to \ell_4^{12}$. □

In such a way one can to interpret a whole series of another classical identities, see [15], [36], [44].

Let us consider an important generalization of Example 1.

EXAMPLE 3 (see [36]) The identity

$$\frac{1}{n}\sum_{k=1}^{n}\left(\xi_1 \cos\frac{\pi k}{n} + \xi_2 \sin\frac{\pi k}{n}\right)^p = \frac{p!}{2^p(\frac{p}{2})!^2}(\xi_1^2 + \xi_2^2)^{\frac{p}{2}}, \quad n = \frac{p}{2} + 1, \tag{4}$$

defines an isometric embedding $\ell_2^2 \to \ell_p^n$. Moreover, in this case n is the minimal possible for given p, so that

$$N_{\mathbf{R}}(2,p) = \frac{p}{2} + 1. \tag{5}$$

In the independent works [36] and [44] an equivalence between isometric embeddings of real spaces $\ell_2^m \to \ell_p^n$ and cubature formulas on the unit sphere $\mathbf{S}^{m-1} \subset \ell_2^m$ was established and some lower bounds for $N_{\mathbf{R}}(m,p)$ were obtained on this base. In addition, a group orbits method for constructing of isometric embeddings was developed in [36]. For cubature formulas such a method comes back to Ditkin and Ljusternik [17] and Sobolev [48] and was widely applied (see [2], [12], [21], [45], [46], [49]) in order to construct cubature formulas equal weights (*the cubature formulas of Chebyshev type* or, equivalently, *the spherical designs*). The concept of spherical designs was introduced in [14], the paper of Delsarte, Goethals and Seidel containing a series of important examples and fundamental bounds. The problem of existence of spherical designs was in general open until [47]. Some further constructions were done in [5], [6], [7], [24], [31], [39], [45] and other works. The theory of general cubature formulas was initiated by Radon [43] and continued by Stroud [51] and Mysovskikh, see references in [41]. Now it is a developed subject, see [41], [53], [58], [59], [60].

The problem of isometric embeddings of complex spaces $\ell_2^m \to \ell_p^n$ was considered by König in [29], where the upper and lower bounds for $N_{\mathbf{C}}(m,p)$ were obtained and some concrete isometric embeddings over $\mathbf{C}$ and $\mathbf{R}$ were constructed using cubature formulas and some other sources.

In the present work we investigate the isometric embeddings $\ell_2^m \to \ell_p^n$ with even p (a necessary condition) over all classical fields, namely, the real field $\mathbf{R}$, the complex field $\mathbf{C}$ and the quaternionic field $\mathbf{H}$. The quaternionic isometric embeddings $\ell_2^m \to \ell_p^n$ are considered here for the first time. This case is more complicated since the field $\mathbf{H}$ is noncommutative.

We denote the basis field by $\mathbf{K}$ and develop a unified theory where the differences between concrete fields are reduced to a minimum. An essential ingredient of this theory is that the above mentioned equivalence between isometric embeddings and cubature formulas can be generalized to any $\mathbf{K}$. In turn, a unified theory of designs over all classical fields was done by Neumaier [42], see also [3], [11], [20], [28], [32]). We show that the cubature formulas with arbitrary weights can be completely included in these frameworks.

Later on all considered linear spaces over $\mathbf{K}$ are supposed to be right ones, otherwise the opposite is stated.

We denote by $\ell_{p;\mathbf{K}}^n$ the right linear space $\mathbf{K}^n$ provided with the norm

$$\|x\|_p = \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{\frac{1}{p}}, \tag{6}$$

where ξ_k are the canonical coordinates, so that x is the column $[\xi_k]_1^n$. (If $\mathbf{K}$ is a fixed a priori, we write ℓ_p^m instead of $\ell_{p;\mathbf{K}}^m$ for short.) The space $\ell_{2;\mathbf{K}}^m$ is an Euclidean space with respect to the inner product

$$\langle x, y\rangle = \sum_{k=1}^{m} \overline{\xi}_k \eta_k \tag{7}$$

where $x = [\xi_k]_1^m$ and $y = [\eta_k]_1^m$. Let us stress that $\langle x, y\rangle$ is a $\mathbf{K}$-linear functional with respect to y while

$$\langle x\gamma, y\rangle = \overline{\gamma}\langle x, y\rangle, \quad \gamma \in \mathbf{K}. \tag{8}$$

In what follows we denote by $\mathbf{S}(E)$ the unit sphere of an Euclidean space E. In fact, we only consider $E = \mathbf{K}^m$ provided with an inner product, as a rule, $E = \ell_{2;\mathbf{K}}^m$.

All results about isometric embeddings $\ell_{2;\mathbf{K}}^m \to \ell_{p;\mathbf{K}}^n$ are collected in Section 4.7 being consequences of the theory of cubature formulas developed in Chapter 4. In this context the crucial role belongs to the cubature formulas for polynomial functions on projective spaces. The point is that the functions of form

$$x \mapsto |\langle x, y\rangle|^p, \quad (x \in \mathbf{S}(\mathbf{K}^m), \quad y \in \mathbf{K}^m) \tag{9}$$

are projectively invariant, i.e. they are invariant with respect to the multiplication

$$x \mapsto x\gamma, \quad |\gamma| = 1. \tag{10}$$

This means that the functions (9) are well-defined on the projective space $\mathbf{KP}^{m-1}$ which can be realized as the quotient space of the unit sphere $\mathbf{S}(\mathbf{K}^m)$ under the multiplicative action (10) of the group

$$U(\mathbf{K}) = \{\gamma : \gamma \in \mathbf{K}, \ |\gamma| = 1\}. \tag{11}$$

The *polynomial functions* on $\mathbf{KP}^{m-1}$ are defined as the $U(\mathbf{K})$-invariant restrictions to $\mathbf{S}(\mathbf{K}^m)$ of *complex-valued polynomials* initially given on the realification

$$\mathbf{R}^{\delta m} \equiv \mathbf{K}^m, \quad \delta = [\mathbf{K} : \mathbf{R}]. \tag{12}$$

The polynomial functions are in $1-1$ correspondence (by restriction to $\mathbf{S}(E)$) with the homogeneous polynomials (forms) ϕ of even degree $2t$ such that

$$\phi(x\gamma) = \phi(x)|\gamma|^{2t}, \quad \gamma \in \mathbf{K}. \tag{13}$$

Conversely, for any complex-valued polynomial ϕ on $\mathbf{R}^{\delta m}$ with property (13) its restriction to $\mathbf{S}(E)$ is a polynomial function on $\mathbf{KP}^{m-1}$. In particular, (9) are the *elementary polynomial functions.*

We denote by $\mathtt{Pol}_{\mathbf{K}}(E; 2t)$ the space of all polynomial functions of degree $2t$ on E. The fundamental property of this space is the orthogonal decomposition

$$\mathtt{Pol}_{\mathbf{K}}(E; 2t) = \mathtt{Harm}_{\mathbf{K}}(E; 0) \oplus \mathtt{Harm}_{\mathbf{K}}(E; 2) \oplus \ldots \oplus \mathtt{Harm}_{\mathbf{K}}(E; 2t), \tag{14}$$

where $\mathtt{Harm}_{\mathbf{K}}(E; 2k)$ is the space of harmonic polynomial functions which are the restrictions of harmonic forms of degree $2k$ satisfying (13). The space of such forms will be denoted by $\mathcal{H}_{\mathbf{K}}(E; 2k)$. The orthogonality in (14) corresponds to the standard inner product

$$(\phi_1, \phi_2) = \int_{\mathbf{S}(E)} \overline{\phi_1}\phi_2 \, d\sigma, \tag{15}$$

where σ is the normalized Lebesgue measure on the unit sphere $\mathbf{S}(E)$.

It is important to know what is the dimension of $\mathtt{Pol}_{\mathbf{K}}(E; 2t)$. The problem is easy for $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ but the case $\mathbf{K} = \mathbf{H}$ is nontrivial, see [25]. In our context we have
COROLLARY 3.3.5 (cf. (3.145)) [2]. *The following formula holds*

$$\dim \mathtt{Pol}_{\mathbf{K}}(E; 2t) = \Lambda_{\mathbf{K}}(m, 2t), \tag{16}$$

[2] All statements in the Introduction are enumerated as in the further text. Some formulations in the Introduction are slightly modified.

where

$$\Lambda_{\mathbf{K}}(m,2t)=\begin{cases}\binom{m+2t-1}{m-1} & (\mathbf{K}=\mathbf{R})\\[2ex] \binom{m+t-1}{m-1}^2 & (\mathbf{K}=\mathbf{C})\\[2ex] \frac{1}{2m-1}\binom{2m+t-2}{2m-2}\cdot\binom{2m+t-1}{2m-2} & (\mathbf{K}=\mathbf{H})\end{cases}. \tag{17}$$

Any non-empty finite set X of projectively distinct points on $\mathbf{S}(E)$ is called a *projective code.* Its *angle set* is defined as

$$a(X)=\{2|\langle x,y\rangle|^2-1:\ x,y\in X,\quad x\neq y\}\subset[-1,1). \tag{18}$$

For a non-empty finite subset $Y\subset\mathbf{S}(E)$ its *projectivization* can be defined as a projective code X such that the natural images of X and Y in $\mathbf{KP}^{m-1}$ coincide.

A *projective cubature formula of index* $2t$ is an identity

$$\int\phi\,d\varrho=\int_{\mathbf{S}(E)}\phi\,d\sigma,\quad \phi\in\mathtt{Pol}_{\mathbf{K}}(E;2t), \tag{19}$$

where ϱ is a finitely supported measure such that the support

$$\mathrm{supp}\varrho=\{x_k\}_1^n\subset\mathbf{S}(E) \tag{20}$$

is a projective code.The points $x_1,\dots,x_n$ are called the *nodes* and their measures $\varrho_k=\varrho(x_k),\quad 1\leq k\leq n$, are called the *weights.* The support of the projective cubature formula of index $2t$ in the case of equal weights is called a *projective design of index* $2t$ or a *projective* $2t$*-design.*[3]

In particular, the cubature formula (19) can be applied to the elementary polynomial functions with $p=2t$. As a result,

$$\sum_{k=1}^n|\langle x_k,x\rangle|^p\varrho_k=\int_{\mathbf{S}(E)}|\langle y,x\rangle|^p\,d\sigma(y),\quad x\in E. \tag{21}$$

The latter integral is an unitary invariant polynomial function of x of degree p. All such functions are $\mathrm{const}\cdot\langle x,x\rangle^{\frac{p}{2}}$. In the case (21) the constant factor is

$$\int_{\mathbf{S}(E)}|\eta_1|^p\,d\sigma(y)\equiv\frac{1}{\Upsilon_{\mathbf{K}}\left(m,\frac{p}{2}\right)} \tag{22}$$

where η_1 is the first canonical coordinate of y. Thus, we have the following *Hilbert identity*

$$\Upsilon_{\mathbf{K}}\left(m,\frac{p}{2}\right)\int_{\mathbf{S}(E)}|\langle y,x\rangle|^p\,d\sigma(y)=\langle x,x\rangle^{\frac{p}{2}},\quad x\in E. \tag{23}$$

The constant $\Upsilon_{\mathbf{K}}\left(m,\frac{p}{2}\right)$ can be explicitly calculated, see Chapter 2, Corollary 2.2.5.

[3]"t-design in projective space" according to [28].

Comparing (21) to (23) we obtain

$$\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right) \sum_{k=1}^{n} |\langle x_k, x\rangle|^p \varrho_k = \langle x, x\rangle^{\frac{p}{2}}, \tag{24}$$

whence

$$\sum_{k=1}^{n} |\langle u_k, x\rangle|^p = \langle x, x\rangle^{\frac{p}{2}}, \tag{25}$$

where

$$u_k = \left(\varrho_k \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)\right)^{\frac{1}{p}} x_k, \quad 1 \le k \le n. \tag{26}$$

Like the Examples $1-3$, the identity (25) means that the mapping

$$x \mapsto \begin{bmatrix} \langle u_1, x\rangle \\ \vdots \\ \langle u_n, x\rangle \end{bmatrix} \tag{27}$$

is an isometric embedding $\ell^m_{2;\mathbf{K}} \to \ell^n_{p;\mathbf{K}}$. We call (25) the *basis identity.*

In fact, any isometric embedding $f : \ell^m_{2;\mathbf{K}} \to \ell^n_{p;\mathbf{K}}$ can be obtained as before. Indeed,

$$fx = \sum_{k=1}^{n} \zeta_k(x) v_k \tag{28}$$

where $(v_k)_1^n$ is the canonical basis in $\ell^n_{p;\mathbf{K}}$ and $\zeta_k(x)$ are the canonical coordinates. The coefficients $\zeta_k(x)$ are the $\mathbf{K}$-linear functionals on the Euclidean space $\ell^m_{2;\mathbf{K}}$. By the Riesz theorem (which is true despite of noncommutativity of $\mathbf{K}$) we have $\zeta_k(x) = \langle u_k, x\rangle$, where $u_k \in \ell^m_{2;\mathbf{K}}$, $1 \le k \le n$. The system $(u_k)_1^n$ is called the *frame* of an isometric embedding $\ell^m_{2;\mathbf{K}} \to \ell^n_{p;\mathbf{K}}$. As a result,

$$fx = \sum_{k=1}^{n} \langle u_k, x\rangle v_k, \tag{29}$$

hence, the basis identity (25) is the coordinate form of the isometry property $\|fx\|_p = \|x\|_2$.
THEOREM 4.7.1. *An isometric embedding $f : \ell^m_{2;\mathbf{K}} \to \ell^n_{p;\mathbf{K}}$ exists if and only if there exists a projective cubature formula of index p with some number $\nu \le n$ of nodes on $\mathbf{S}(\mathbf{K}^m)$.*

We have already explained how this theorem can be proven in the "if" direction. The proof in the "only if" direction is more profound being based on the `Addition Theorem` for harmonic polynomial functions. The latter states that for any orthonormal basis $(s_{kj})_{j=1}^{\hbar_{m,2k}}$ in $\mathtt{Harm}_{\mathbf{K}}(E; 2k)$ the `Addition Formula` holds:

$$\sum_{j=1}^{\hbar_{m,2k}} \overline{s_{kj}(x)} s_{kj}(y) = q_{m,k}(|\langle x, y\rangle|^2) \quad (x, y \in \mathbf{S}(E)), \tag{30}$$

where $q_{m,k}(u)$ is a polynomial of degree k. In fact,

$$q_{m,k}(u) = b^{(\delta)}_{m,k} P_k^{(\alpha,\beta)}(2u - 1), \tag{31}$$

where

$$\alpha = \frac{\delta m - \delta - 2}{2}, \quad \beta = \frac{\delta - 2}{2} \tag{32}$$

and $P_k^{(\alpha,\beta)}(u)$ is the Jacobi polynomial. (See (3.127) for explicit expression of the coefficient $b_{m,k}^{(\delta)}$). Such a developed form of the Addition Theorem is classical in the real case, however, in the complex case it was proven recently by Koornwinder [30]. For the quaternionic case the result is due to Hoggar [25]. Both authors used the group representation theory. Our proof in Section 3.3 is elementary, based only on the classical theory of spherical harmonics. Let us formulate a series of theorems about existence of projective cubature formulas (Section 4.3). Respectively, there are the existence theorems for isometric embeddings $\ell_{2;\mathbf{K}}^m \to \ell_{p;\mathbf{K}}^n$ (Section 4.7).

THEOREM 4.4.2. *For any* $t \in \mathbf{N},\ t \geq 1$, *there exists a projective cubature formula of index* $2t$ *with* n *nodes, where*

$$n \leq \Lambda_{\mathbf{K}}(m, 2t). \tag{33}$$

The result follows from the Hilbert identity (23) which shows that the polynomial

$$\left(\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)\right)^{-1} \langle x, x\rangle^{\frac{p}{2}} \tag{34}$$

belongs to the closed convex hull of the elementary polynomial functions. By Caratheodory's theorem this polynomial is a convex combination of n polynomial functions,

$$n \leq \dim \mathtt{Pol}_{\mathbf{K}}(E; 2t) + 1 = \Lambda_{\mathbf{K}}(m, 2t) + 1. \tag{35}$$

This yields the basis identity (25) with some $n \leq \Lambda_{\mathbf{K}}(m, 2t) + 1$. In fact, we have (33) in view of some additional arguments. Moreover, (33) can be reduced to $n \leq \Lambda_{\mathbf{K}}(m, 2t) - 1$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}$, see [13].

Note that Theorem 4.4.2 is a projective counterpart of Tchakaloff's Theorem [55] on existence of cubature formulas on compact subsets in $\mathbf{R}^m$, see also [41].

For given $m = \dim E$ and $t \in \mathbf{N},\ t \geq 1$ a projective cubature formula of index $2t$ is called *minimal* if the number of nodes is minimal possible. Thus, the minimal number of nodes of a projective cubature formula of index $p = 2t$ on $\mathbf{S}(\mathbf{K}^m)$ coincides with $N_{\mathbf{K}}(m, p)$, the minimal number n such that an isometric embedding $\ell_{2;\mathbf{K}}^m \to \ell_{p;\mathbf{K}}^n$ does exist. (See Corollary 4.7.8.)

In order to improve the upper bound (33) for $N_{\mathbf{K}}(m, p)$ we have developed an inductive (on m) construction of projective cubature formulas. This approach results in the following

THEOREM 4.4.7. *Each real projective cubature formula of index* $2t$ *with* n *nodes on* $\mathbf{S}(\mathbf{K}^{m-1}) \equiv \mathbf{S}(\mathbf{R}^{\delta m - \delta})$ *generates a* $\mathbf{K}$*-projective cubature formula of the same index* $2t$ *with* N *nodes on* $\mathbf{S}(\mathbf{K}^m)$ *where*

$$N = (t+1) N_{\mathbf{R}}\left(\delta, 2\left[\frac{t}{2}\right]\right) n. \tag{36}$$

A direct corollary from Theorem 4.4.7 is

THEOREM 4.4.8. *The inequality holds:*

$$N_{\mathbf{K}}(m, 2t) \leq (t+1) N_{\mathbf{R}}\left(\delta, 2\left[\frac{t}{2}\right]\right) N_{\mathbf{R}}((m-1)\delta, 2t). \tag{37}$$

Note that the factor $t+1$ comes from the Gauss-Jacobi quadrature formula.

The result for the isometric embeddings which corresponds to the Theorem 4.4.7 is

THEOREM 4.7.16. *Each isometric embedding* $\ell_{2;\mathbf{R}}^{\delta(m-1)} \to \ell_{p;\mathbf{R}}^{n}$ *generates an isometric embedding* $\ell_{2;\mathbf{K}}^{m} \to \ell_{p;\mathbf{K}}^{N}$ *where*

$$N = \left(\frac{p}{2}+1\right) N_{\mathbf{R}}\left(\delta, 2\left[\frac{p}{4}\right]\right)\nu, \quad \nu \le n. \tag{38}$$

The factor ν in (38) depends on the initial embedding.

The Theorems 4.4.7 and 4.7.16 allow us to obtain some new projective cubature formulas (isometric embeddings) over $\mathbf{K}$ using some known ones over $\mathbf{R}$. In the real case this yields

COROLLARY 4.4.12. *Assume that for given* m, t *there exists a real projective cubature formula of index* $2t$ *with* n *nodes on* $\mathbf{S}^{m-1}$. *Then there exists a real projective cubature formula of the same index* $2t$ *with*

$$N = (t+1)^{M-m} n \tag{39}$$

nodes on $\mathbf{S}^{M-1}$, $M \ge m$.

Respectively, we have

THEOREM 4.7.18. *Assume that for given* m, p *there exists an isometric embedding* $\ell_{2;\mathbf{R}}^{m} \to \ell_{p;\mathbf{R}}^{n}$. *Then for each* $M \ge m$ *there exists an isometric embedding* $\ell_{2;\mathbf{R}}^{M} \to \ell_{p;\mathbf{R}}^{N}$ *with*

$$N = \left(\frac{p}{2}+1\right)^{M-m}\nu, \quad \nu \le n. \tag{40}$$

Corollary 4.4.12 allows us to reduce the upper bound (33) in some cases. For example,

$$N_{\mathbf{R}}(m, 2t) \le (t+1)^{m-1}, \; m \le 4. \tag{41}$$

(See also Corollaries 4.4.22, 4.4.24.) For $m > 4$ the bound (41) is valid but this is worse than (33) yields. Note that (41) also follows from a result on product cubature formulas, see [53].

In order to apply Theorem 4.4.7 to the complex case we note that the factor $N_{\mathbf{R}}\left(2, 2\left[\frac{t}{2}\right]\right)$ in (37) is known by (5). Thus, we have

THEOREM 4.4.16. *Assume that for given* m, t *there exists a real projective cubature formula of index* $2t$ *with* n *nodes on* $\mathbf{S}^{2m-3}$. *Then there exists a complex projective cubature formula of the same index* $2t$ *with*

$$N = (t+1)\left(\left[\frac{t}{2}\right]+1\right)n \tag{42}$$

nodes on $\mathbf{S}(\mathbf{C}^m)$.

THEOREM 4.7.19. *Assume that for given* m, p *there exists an isometric embedding* $\ell_{2;\mathbf{R}}^{2m} \to \ell_{p;\mathbf{R}}^{n}$. *Then there exists an isometric embedding* $\ell_{2;\mathbf{C}}^{m+1} \to \ell_{p;\mathbf{C}}^{N}$ *with*

$$N = \left(\frac{p}{2}+1\right)\left(\left[\frac{p}{4}\right]+1\right)\nu, \quad \nu \le n. \tag{43}$$

For the quaternionic case we need to consider the factor $N_{\mathbf{R}}\left(4, 2\left[\frac{t}{2}\right]\right)$ appearing in (36) and (37). The exact values of that are mostly unknown at present. The exceptions are only

$$N_{\mathbf{R}}(4,2) = 4, \quad N_{\mathbf{R}}(4,4) = 11. \tag{44}$$

The first one is trivial, the second one is due to Stroud [52] in a context of cubature formulas.

The inequality (41) yields

$$N_{\mathbf{R}}\left(4, 2\left[\frac{t}{2}\right]\right) \leq \left(\left[\frac{t}{2}\right] + 1\right)^3 \tag{45}$$

for all t. According to (37), we have

THEOREM 4.4.17. *The inequality*

$$N_{\mathbf{H}}(m, 2t) \leq (t+1)\left(\left[\frac{t}{2}\right] + 1\right)^3 N_{\mathbf{R}}(4(m-1), 2t) \tag{46}$$

holds.

For $t < 20$, $t \neq 12, 13$, the coefficient $(t+1)\left(\left[\frac{t}{2}\right]+1\right)^3$ in (46) can be improved. The point is that some upper bounds for $N_{\mathbf{R}}(4, 2k)$, $2 \leq k \leq 9$, follow from known results.

THEOREM 4.4.20. *Assume that for given m, t there exists a real projective cubature formula of index $2t$ with n nodes on $\mathbf{S}^{4m-5}$. Then there exists a quaternionic projective cubature formula of the same index $2t$ with*

$$N = \begin{cases} L(t)n, & 2 \leq t < 20, \quad t \neq 12, 13 \\ (t+1)\left(\left[\frac{t}{2}\right]+1\right)^3 n, & t \geq 20 \text{ or } t = 12, 13 \end{cases} \tag{47}$$

nodes on $\mathbf{S}(\mathbf{H}^m)$, the function $L(t)$ is defined by the tables (4.166) and (4.167).

THEOREM 4.7.20. *Assume that for given m, p there exists an isometric embedding $\ell_{2;\mathbf{R}}^{4m} \to \ell_{p;\mathbf{R}}^{n}$. Then there exists an isometric embedding $\ell_{2;\mathbf{H}}^{m+1} \to \ell_{p;\mathbf{H}}^{N}$ with*

$$N = \begin{cases} L(\frac{p}{2})\nu, & 4 \leq \frac{p}{2} < 40, \quad p \neq 24, 26 \\ (\frac{p}{2}+1)\left(\left[\frac{p}{4}\right]+1\right)^3 \nu, & p \geq 40 \text{ or } p = 24, 26 \end{cases} \tag{48}$$

and $\nu \leq n$.

In Section 4.4 one can also find Theorems 4.4.21, 4.4.23, 4.4.25, 4.4.27, 4.4.29 and 4.4.31 containing many new concrete projective cubature formulas, for example,

THEOREM 4.4.29. *There exists a quaternionic projective cubature formula of index 10 with 6486480 nodes on $\mathbf{S}(\mathbf{H}^7)$.*

The corresponding isometric embeddings $\ell_{2;\mathbf{K}}^{m} \to \ell_{p;\mathbf{K}}^{n}$ are in the tables (4.332) – (4.338). These results are the best known at present.

The following statement based on Addition Theorem is the main key to obtain a lower bound for $N_{\mathbf{K}}(m, 2t)$.

LEMMA 4.3.9.*Any projective cubature formula of index $2t$ is equivalent to the identity*

$$\sum_{x,y} f(2|\langle x, y\rangle|^2 - 1)\varrho(x)\varrho(y) = \int_{-1}^{1} f(u)\Omega_{\alpha,\beta}(u)\, du, \quad f \in \Pi_t, \tag{49}$$

where Π_t is the space of real polynomials of degree t and $\Omega_{\alpha,\beta}(u)$ is the normalized Jacobi weight,

$$\Omega_{\alpha,\beta}(u) = \frac{(1-u)^\alpha (1+u)^\beta}{\int_{-1}^{1}(1-u)^\alpha (1+u)^\beta \, du}. \tag{50}$$

COROLLARY 4.3.13. *Let a projective cubature formula of index $2t$ be valid. Then*

$$n \geq \frac{f(1)}{\displaystyle\int_{-1}^{1} f(u)\Omega_{\alpha,\beta}(u)\,du} \tag{51}$$

for any real polynomial $f \in \Pi_t$ such that $f(1) > 0$ and $f|a(X) \geq 0$.

As a consequence, the following linear programming problem arises:

$$\begin{cases} \Psi \in \Pi_t, \quad \Psi(u) \geq 0 \quad (-1 \leq u \leq 1), \\ \displaystyle\int_{-1}^{1} \Psi(u)\Omega_{\alpha,\beta}(u)du = 1, \\ \Psi(1) \to \max, \end{cases} \tag{52}$$

The problem was solved by Szegö in [54]. The answer is

$$\Psi_{\max}(u) = (1+u)^\varepsilon \left(P_{[\frac{t}{2}]}^{(\alpha+1,\beta+\varepsilon)}(u) \right)^2 \tag{53}$$

up to proportionaliy.

THEOREM 4.3.15. *The inequality*

$$N_{\mathbf{K}}(m, 2t) \geq \Lambda_{\mathbf{K}}(m,t) \tag{54}$$

holds.

The important question about exactness of this lower bound arises. A projective cubature formula (an isometric embedding) is called *tight* if $n = \Lambda_{\mathbf{K}}(m,t)$. For example, any isometric embedding $\ell_2^2 \to \ell_p^{\frac{p}{2}+1}$ over $\mathbf{R}$ is tight by (5). Another known tight isometric embeddings over $\mathbf{R}$ are

$$\ell_2^3 \to \ell_4^6, \quad \ell_2^7 \to \ell_4^{28}, \quad \ell_2^8 \to \ell_6^{120}, \quad \ell_2^{23} \to \ell_4^{276}, \quad \ell_2^{23} \to \ell_6^{2300}, \quad \ell_2^{24} \to \ell_{10}^{98280}, \tag{55}$$

see [36], [44]. Over $\mathbf{C}$ the known tight isometric embeddings are

$$\ell_2^2 \to \ell_4^4, \quad \ell_2^2 \to \ell_6^6, \quad \ell_2^3 \to \ell_4^9, \quad \ell_2^4 \to \ell_6^{40}, \quad \ell_2^6 \to \ell_6^{126}, \quad \ell_2^8 \to \ell_4^{64}, \tag{56}$$

see [29]. As aforesaid, the problem of isometric embeddings over $\mathbf{H}$ was not considered earlier.

THEOREM 4.3.18. *If a projective cubature formula of index $2t$ is tight then*

(i) *the weights are equal, i.e. its support X is a projective design;*

(ii) *with $\varepsilon = \mathrm{res}(t) (\mathrm{mod}\ 2)$ the polynomial*

$$(1+u)^\varepsilon P_{[\frac{t}{2}]}^{(\alpha+1,\beta+\varepsilon)}(u) \tag{57}$$

annihilates the angle set $a(X)$.

Conversely, let X be a projective code such that

$$|X| = \Lambda_{\mathbf{K}}(m,t) \tag{58}$$

and let the angle set $a(X)$ be annihilated by the polynomial (57). Then X is a tight projective $2t$-design.

Respectively, we have

THEOREM 4.7.14. *Let*

$$\mathcal{R}_{\mathbf{K}}(m,p) = \left(\frac{\Upsilon_{\mathbf{K}}\left(m,\frac{p}{2}\right)}{\Lambda_{\mathbf{K}}\left(m,\frac{p}{2}\right)}\right)^{\frac{1}{p}}. \tag{59}$$

If an isometric embedding $f : \ell_2^m \to \ell_p^n$ is tight then

(i) *its frame $(u_j)_1^n$ lies on a sphere $\mathbf{S}_{\mathbf{K}}(m,p)$ of radius $\mathcal{R}_{\mathbf{K}}(m,p)$ centered at origin;*

(ii) *with $\varepsilon = \operatorname{res}(\frac{p}{2})(\mathrm{mod}\ 2)$ the polynomial*

$$(1+u)^{\varepsilon} P^{(\alpha+1,\beta+\varepsilon)}_{[\frac{p}{4}]}(u) \tag{60}$$

annihilates the angle set of the normalized frame

$$\widehat{u}_j = (\mathcal{R}_{\mathbf{K}}(m,p))^{-1} u_j, \quad 1 \le j \le n. \tag{61}$$

Conversely, with

$$n = \Lambda_{\mathbf{K}}\left(m, \frac{p}{2}\right), \tag{62}$$

let a system $(u_j)_1^n \subset \mathbf{K}^m$ lie on the sphere $\mathbf{S}_{\mathbf{K}}(m,p)$ and (**ii**) *holds. Then $(u_j)_1^n$ is the frame of a tight isometric embedding.*

The frame of the tight isometric embedding $\ell_2^2 \to \ell_p^{\frac{p}{2}+1}$ is a "half" of the regular $(p+2)$-gone, see (4). In all tight situations (55) and (56) the frames are the relevant parts of some regular polytopes or some other very symmetric configurations of points on the spheres. For example, the isometric embedding $\ell_2^3 \to \ell_4^6$ comes from the icosahedron (see [36], [44]; the corresponding cubature formula was discovered in [17]). The most effective way to get the symmetric constructions is to take a finite subgroup G of the unitary group $U(\mathbf{K}^m)$ and obtain the support of a cubature formula (the frame of an isometric embedding) as a finite G-invariant set (a G-orbit or the union of some orbits). In this context the index $p = 2t$ depends on properties of G-*invariant* harmonic polynomial functions, see [2], [21], [36]. We extend the method of harmonic invariants to the complex and quaternionic situations.

We say that a projective cubature formula (19) is called G-*invariant* if the measure ϱ is G-invariant. The latter means that $\operatorname{supp}\varrho$ is G-invariant and the function $x \mapsto \varrho(x)$ is constant on any orbit Gx_0, $x_0 \in \operatorname{supp}\varrho$. Also we define the space $\mathcal{H}_{\mathbf{K};G}(E;2k)$ of those $U(\mathbf{K})$-invariant harmonic forms of degree $2k$ which are G-invariant.

PROPOSITION 4.5.12. *Let X be a G-invariant projective code and let ϱ be a G-invariant measure such that $\operatorname{supp}\varrho = X$. Then ϱ defines an (automatically G-invariant) projective cubature formula of index $2t$ if and only if the system of equalities*

$$\int \phi\, d\varrho = \sum_x \phi(x)\varrho(x) = 0, \quad \phi \in \mathcal{H}_{\mathbf{K};G}(E;2k), \quad 1 \le k \le t, \tag{63}$$

holds and, in addition,

$$\int d\varrho = \sum_x \varrho(x) = 1. \tag{64}$$

The following Corollaries are especially important for our purposes.

COROLLARY 4.5.14. *The projectivization of the G-orbit of a point $x_0 \in \mathbf{S}(E)$ is a projective $2t$-design if and only if all forms from $\mathcal{H}_{\mathbf{K};G}(E;2k)$, $1 \le k \le t$, vanish at the point x_0.*

COROLLARY 4.5.15. *Let X be a G-invariant projective code. Let t be a positive integer such that*

$$\mathcal{H}_{\mathbf{K};G}(E;2k) = 0, \quad 1 \le k \le t. \tag{65}$$

Then X is a projective $2t$-design.

In Section 4.6 we systematically apply these criterions to obtain the projective cubature formulas on $\mathbf{S}(\mathbf{C}^2)$ based on orbits of finite subgroups of $SU(2)$. The corresponding results are presented in [35]. All those subgroups are known, see [50]. In terms of isometric embeddings over $\mathbf{C}$ we obtain

The isometric embedding	The subgroup $G \subset SU(2)$
$\ell_2^2 \to \ell_4^4$	binary dihedral group $\mathcal{D}_2$
$\ell_2^2 \to \ell_6^6$	binary tetrahedral group $\mathcal{T}$
$\ell_2^2 \to \ell_8^{10}$	binary dihedral group $\mathcal{D}_4$
$\ell_2^2 \to \ell_{10}^{12}$	binary icosahedral group $\mathcal{I}$
$\ell_2^2 \to \ell_{12}^{22}$	binary tetrahedral group $\mathcal{T}$
$\ell_2^2 \to \ell_{18}^{60}$	binary icosahedral group $\mathcal{I}$

(66)

All of the above listed isometric embeddings are new except for $\ell_2^2 \to \ell_4^4$ and $\ell_2^2 \to \ell_6^6$ which were found by König [29] in a different way.

We prove that our *isometric embedding $\ell_2^2 \to \ell_8^{10}$ is minimal but not tight.*

A very interesting example is $\ell_2^2 \to \ell_{10}^{12}$. *The frame X of this isometric embedding is a tight projective 10-design.* This yields the affirmative answer to a question from [4] and disproves a statement from [27]. The corresponding angle set contains some *irrational* numbers, namely,

$$a(X) = \left\{ -\frac{\sqrt{5}}{5}, -1, \frac{\sqrt{5}}{5} \right\}. \tag{67}$$

Thus, we obtain a counterexample to a theorem from [42], see also Theorem 1.9 in [4].

The Chapters 1 −3 contain an algebraic and analytic background.

Chapter 1 is devoted to the *noncommutative* linear algebra as a necessary tool for the quaternionic case. Artin's monograph [1] and Bourbaki's volume [9] are standard sources for the subject but we present it in a self-contained form for the reader convenience.

In the first Sections of Chapter 1 the basis field $\mathbf{K}$ is arbitrary. Starting with Section 1.9 we restrict the class of admissible fields and, eventually, we consider the classical fields only. One of central facts in the noncommutative linear algebra we need is the following Witt Theorem [57].

THEOREM 1.9.32. *Let E be an Euclidean space. The systems $(v_k)_1^\nu$ and $(u_k)_1^\nu$ are unitary equivalent if and only if their Gram matrices coincide.*

The "only if" part of this theorem is trivial but the "if" part is rather deep, see [16].

In Chapter 1 (Section 1.10) we also prove a necessary condition for existence of isometric embedding $\ell_q^m \to \ell_p^n$, $1 < q, p < \infty$. This theorem for $\mathbf{K} = \mathbf{R}$ was proven in [36].

THEOREM 1.10.4. *Let $\mathbf{K} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. Suppose that q, p are distinct and finite, $m \geq 2$. If there exists an isometric embedding $\ell_q^m \to \ell_p^n$ then $q = 2$ and p is an even integer.*

Chapter 2 contains some auxiliary results of analytical nature.

In Section 2.1 we collect a necessary information on the theory of Jacobi polynomials in view of their close relations to the spherical harmonics.

In Section 2.2 some integration formulas for zonal functions on spheres are derived. These formulas effectively work in further applications.

In Section 2.3 we obtain some partial differential equations related to the elementary polynomial functions. In particular, some formulas for the the Laplacian are prepared to prove the Addition Theorem and some other statements.

In Chapter 3 we develop the polynomial function theory.

In Section 3.1 we expose the theory of classical spherical harmonics in a convenient for our purposes form, cf. [40]. The Addition Theorem is the central result of this part of work. The space of the spherical harmonics of degree d is denoted by $\mathtt{Harm}(E; d)$, $E = \mathbf{R}^m$. Here d is an arbitrary integer, i.e. d can be odd in contrast to the projective situation. The space $\mathtt{Harm}(E; d)$ is invariant with respect to the natural action (representation) of the orthogonal group:

$$(g\phi)(x) = \phi(gx), \quad g \in O(m). \tag{68}$$

The Addition Theorem yields the orthogonality

$$\mathtt{Harm}(E; k) \perp \mathtt{Harm}(E; l), \quad k \neq l, \tag{69}$$

taking into account the orthogonality of Jacobi polynomials. (Note that (69) is an important fact of the unitary group representations theory where it follows from the irreducibility of the representation (68) in $\mathtt{Harm}(E; k)$.) Another consequence is an important identity, see Lemma 3.1.22.

In Section 3.2 the polynomial functions on the projective spaces are considered. Note that the space $\mathtt{Pol}_{\mathbf{K}}(E; 2k)$ is the $U(\mathbf{K})$-invariant part of the corresponding space $\mathtt{Pol}_{\mathbf{R}}(E_{\mathbf{R}}; 2k)$ on the real unit sphere $\mathbf{S}(E_{\mathbf{R}})$. (Here $E_{\mathbf{R}}$ is the realification of the space E.) This allows us to use the results of Section 3.2 in the projective situation.

Section 3.3 contains the proof of the Projective Addition Theorem and some its important consequences, the dimension formula (16) and the following `Completeness Theorem` among them.

THEOREM 3.3.7. *In the space $\mathtt{Pol}_{\mathbf{K}}(E; 2t)$ the system of elementary polynomial functions of degree $2t$ is complete,*

$$\mathtt{Pol}_{\mathbf{K}}(E; 2t) = \mathrm{Span}\{|\langle \cdot, y \rangle|^{2t} : y \in E\}. \tag{70}$$

In Section 4.1 of Chapter 4 we expose the theory of real spherical (not projective, in general) cubature formulas of indices $d \in \mathbf{N}$ guided by [14]. This is a good prototype for the above described theory of projective cubature formulas.

Chapter 1

Noncommutative linear algebra

Let $\mathbf{K}$ be a field, i.e., an associative (not necessary commutative) ring with unity 1 such that the set of all nonzero elements is a group with respect to the multiplication. The elements from $\mathbf{K}$ are called **scalars**. We denote them by small Greek letters. In the theory we expose below the field $\mathbf{K}$ is supposed to be fixed a priori and it is called the **basis field** in this context.

For our purposes the most important cases are $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (the real numbers, the complex numbers and the quaternions, respectively.). The only case $\mathbf{K} = \mathbf{H}$ from this classical list is not commutative.

Both of $\mathbf{C}$ and $\mathbf{H}$ are canonical extensions of $\mathbf{R}$, so that $\mathbf{R} \subset \mathbf{C}$ and $\mathbf{R} \subset \mathbf{H}$. In such a way $\mathbf{R}$ is identified with the center of the field $\mathbf{H}$. However, there is no intrinsic characterization for $\mathbf{C}$ as a subfield of $\mathbf{H}$: for any $\varepsilon \in \mathbf{H}$ such that $\varepsilon^2 = -1$ the quaternions $\alpha + \beta\varepsilon$ with real α and β form a subfield of $\mathbf{H}$ which is isomorphic to $\mathbf{C}$. `Later on we denote the standard imagine units in` $\mathbf{H}$ `by` $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} = \mathbf{ij}$ `and` $\mathbf{i}$ `as the standard imagine units in` $\mathbf{C}$. The standard realization of the inclusion $\mathbf{C} \subset \mathbf{H}$ corresponds to $\varepsilon = \mathbf{j}$, so that

$$q = \xi + \eta\mathbf{j} \quad (\xi,\ \eta \in \mathbf{C}) \tag{1.1}$$

is the complex form for the quaternions. Note that $\eta\mathbf{j} = \mathbf{j}\overline{\eta}$ since $\mathbf{ij} = -\mathbf{ji}$ and

$$\overline{\xi + \eta\mathbf{j}} = \overline{\xi} - \eta\mathbf{j} \tag{1.2}$$

for the standard involution in $\mathbf{H}$. The multiplication rule in $\mathbf{H}$ over $\mathbf{C}$ is

$$(\xi + \eta\mathbf{j})(\zeta + \omega\mathbf{j}) = (\xi\zeta - \eta\overline{\omega}) + (\xi\eta + \eta\overline{\zeta})\mathbf{j}. \tag{1.3}$$

1.1 Linear spaces. Subspaces

DEFINITION 1.1.1. *A set F provided with linear operations, i.e. with addition $x + y$, $(x, y \in F)$ and right multiplication by scalars $x\alpha$ $(x \in F,\quad \alpha \in \mathbf{K})$, is called a* **right linear (vector) space**[1] *over $\mathbf{K}$, if the following axioms hold:*

[1]Similarly, the concept of the **left linear space** can be introduced and the corresponding theory can be developed. Also one can consider the **two-sided linear spaces** with the additional axiom $(\beta x)\alpha = \beta(x\alpha)$. If the basis field is commutative then any left linear space can be identified with a right linear space by the agreement $\alpha x \equiv x\alpha$.

1. *F is an Abelian group with respect to the addition.*
   Its zero element is denoted by 0 and the inverse to x is denoted by $-x$.

2. *The multiplication by scalars is unital, i.e.,*

$$x \cdot 1 = x,$$

   and associative, i.e.

$$(x\alpha)\beta = x(\alpha\beta),$$

   so that the product $x\alpha\beta$ is independent of the placement of parentheses.

3. *The distributive laws*

$$(x + y)\alpha = x\alpha + y\alpha, \quad x(\alpha + \beta) = x\alpha + x\beta$$

   hold.

The elements of F are called *vectors* and denoted by small Latin letters.

The trivial examples of linear spaces are: $F = 0$ (the space whose only element is $x = 0$); $F = \mathbf{K}$. These spaces are two-sided. A more general example of a two-sided linear space is the space $M_{n,m}(\mathbf{K})$ of all $n \times m$ matrices over $\mathbf{K}$.

The classical fields $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ can be considered as two-sided linear spaces over $\mathbf{R}$ and the same is true for $\mathbf{H}$ over $\mathbf{C}$: for $x \in \mathbf{H}$ and $\alpha \in \mathbf{C}$ the products $x\alpha$ and αx are defined in $\mathbf{H}$ by the above mentioned standard inclusion $\mathbf{C} \subset \mathbf{H}$.

`Further we fix notation` F `for a right linear space over the field` $\mathbf{K}$. If $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ then the space F is called **real, complex** or **quaternionic space** respectively.

Here are some simple consequences of the axioms 1-3.

1. $x\alpha = 0 \Leftrightarrow x = 0$ or $\alpha = 0$;
2. $y = x\alpha \Rightarrow x = y\alpha^{-1} \ \ (\alpha \neq 0)$;
3. $x \cdot (-1) = -x$.

A subset $X \subset F$ is called a **subspace** if it is closed with respect to the addition and multiplication by scalars. The trivial subspaces are $X = 0$ and $X = F$.

Obviously, any subspace is a linear space per se. Its zero element is the same as for the whole space.

The intersection of any family of subspaces is a subspace.

Consider a system (a finite sequence) of vectors $u_1, \ldots, u_l$. A vector x is a **linear combination** of these vectors with scalar coefficients $\alpha_1, \ldots, \alpha_l$ if

$$x = \sum_{i=1}^{l} u_i\alpha_i.$$

In particular, if $\alpha_1 = \ldots = \alpha_l = 0$ then $x = 0$. Such a linear combination is called **trivial.** Obviously, if X is a subspace and $u_1, \ldots, u_l$ belong to X then all their linear combinations belong to X as well.

LEMMA 1.1.2. *If a vector z is a linear combination of some vectors $u_1, \dots, u_l$ and each of u_i is a linear combination of some vectors $v_1, \dots, v_m$, then z is a linear combination of $v_1, \dots, v_m$.*

Proof. Let

$$z = \sum_{i=1}^{l} u_i \beta_i$$

and

$$u_i = \sum_{j=1}^{m} v_j \gamma_{ji}, \ 1 \le i \le l.$$

Then

$$z = \sum_{i=1}^{l} \Big(\sum_{j=1}^{m} v_j \gamma_{ji} \Big) \beta_i = \sum_{j=1}^{m} v_j \Big(\sum_{i=1}^{l} \gamma_{ji} \beta_i \Big) = \sum_{j=1}^{m} v_j \alpha_j,$$

where

$$\alpha_j = \sum_{i=1}^{l} \gamma_{ji} \beta_i.$$

□

The set of all linear combinations of the vectors $u_1, \dots, u_l$ is called a **linear span** of these vectors. It is denoted by $\text{Span}(u_1, \dots, u_l)$. The vectors $u_1, \dots, u_l$ belong to their linear span since

$$u_k = \sum_{i=1}^{l} u_i \delta_{ik}, \quad 1 \le k \le l,$$

where δ_{ik} is Kronecker's delta.

For any nonempty subset $Z \subset F$ the set $\text{Span}(Z)$ is defined as the set of all linear combinations of all finite subsets of Z.

COROLLARY 1.1.3. $\text{Span}(Z)$ *is a subspace for any $Z \subset F$.*

The vectors $u_1, \dots, u_l$ are called **linearly dependent** if a nontrivial linear combination of them is equal to zero. Otherwise, $u_1, \dots u_l$ are called **linearly independent**.

Note that

1. With $l > 1$ a system $(u_1, \dots, u_l)$ is linearly dependent if and only if it contains a vector which is a linear combination of remaining ones. In particular, (u_1, u_2) is linearly dependent if and only if one of them is proportional to other.

2. System (u_1) is linearly independent if and only if $u_1 \ne 0$.

3. If a subsystem of a system of vectors is linearly dependent then the whole system is linearly dependent. Therefore, if a vector in a system is zero then the system is linearly dependent.

4. If a system $(u_1, \dots, u_l)$ is linearly independent and $u_{l+1} \notin \text{Span}(u_1, \dots, u_l)$ then $(u_1, \dots, u_l, u_{l+1})$ is also linearly independent.

5. If at least one of vectors $u_1, \dots, u_l$ is different from zero then either the system $(u_1, \dots, u_l)$ is linearly independent or there exists a linearly independent subsystem $(u_{i_1}, \dots, u_{i_m})$, $m < l$, such that all vectors $u_1, \dots, u_l$ are linear combinations of those ones.

LEMMA 1.1.4. *With $m > l$ any m vectors from* $\mathrm{Span}(u_1, \dots, u_l)$ *are linearly dependent.*

Proof. One can assume that $m = l + 1$ and then prove the lemma by induction on l.

Let $l = 1$. Take two vectors v_1 and v_2 from $\mathrm{Span}(u_1)$. Then for some $\alpha, \beta \in \mathbf{K}$ we have

$$v_1 = u_1\alpha, \quad v_2 = u_1\beta,$$

If $\alpha = 0$, then $v_1 = 0$ hence, v_1 and v_2 are linearly dependent. If $\alpha \neq 0$, then

$$v_2 = u_1\beta = (v_1\alpha^{-1})\beta = v_1(\alpha^{-1}\beta).$$

Hence, v_1 and v_2 are linearly dependent again.

Now suppose that the lemma is true for some $l = m - 1$ $(m > 1)$ and prove it for $l = m$. Let $v_1, \dots, v_{m+1}$ are some $m + 1$ vectors from $\mathrm{Span}(u_1, \dots, u_m)$ say,

$$v_k = \sum_{i=1}^{m} u_i\alpha_{ik}, \quad k = 1, \dots, m + 1.$$

First note that if $\alpha_{1,k} = 0$ for all k, then $v_1, \dots, v_m$ belong to the $\mathrm{Span}(u_2, \dots, u_m)$. By induction, they are linearly dependent. A fortiori, $v_1, \dots, v_m, v_{m+1}$ are linearly dependent.

Now suppose that not all of $\alpha_{m,k}$ are equal to zero. Without loss of generality we can assume that $\alpha_{1,1} \neq 0$. Then

$$v_1\alpha_{1,1}^{-1} = u_1 + \sum_{i=2}^{m} u_i(\alpha_{i,1}\alpha_{1,1}^{-1}).$$

It implies that the vectors

$$w_k = v_k - v_1(\alpha_{1,1}^{-1}\alpha_{1,k}), \quad k = 2, \dots, m + 1,$$

belong to the $\mathrm{Span}(u_2, \dots, u_m)$. By induction the vectors $w_1, \dots, w_m$ are linearly dependent, i.e.

$$\sum_{k=2}^{m+1} w_k\gamma_k = 0$$

for some nontrivial system of coefficients $\gamma_2, \dots, \gamma_{m+1}$. Hence,

$$\sum_{k=1}^{m+1} v_k\gamma_k = 0$$

where

$$\gamma_1 = -\alpha_{1,1}^{-1}\sum_{k=2}^{m+1} \alpha_{1,k}\gamma_k.$$

□

1.2 The bases

A system of vectors is called **complete** if its linear span coincides with the space F. A complete linearly independent system is called a *basis* of F. For example, the systems (1), $(1, \mathbf{i})$, $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ are the basis over $\mathbf{R}$ in $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ respectively, the **canonical bases**.

If the cardinalities of all linearly independent systems in F are bounded from above then the space F is called **finite-dimensional.** In this case the maximal number of linearly independent vectors in F is called the **dimension** of F and denoted by $\dim F$. If the space F is not finite-dimensional then it is called **infinite-dimensional**.

If $F = 0$ then there are no linearly independent vectors in F so, $\dim F = 0$ by definition. Obviously, $\dim \mathbf{K} = 1$.

`In what follows all linear spaces under considerations` (in particular, F) `are suppose to be finite-dimensional` though many of definitions and statements below remain in force for the infinite-dimensional case. `We set` $\dim F = n > 0$ `and fix this notation, so that the space` F `is` n**-dimensional**. Obviously, $\dim X \leq n$ for any subspace $X \subset F$.

THEOREM 1.2.1. *Any system of n linearly independent vectors is a basis.*

Proof. Let $u_1, \ldots, u_n$ be linearly independent. If they do not constitute the basis then there exists a vector $v \notin \text{Span}(u_1, \ldots, u_n)$ hence, the system $(u_1, \ldots u_n, v)$ is linearly independent. This contradicts to the definition of dimension since $\dim F = n$. □

COROLLARY 1.2.2. *There exists a basis consisting of n elements.*

COROLLARY 1.2.3. *For any ν, $0 \leq \nu \leq n$, there exists a subspace $X \subset F$ such that* $\dim X = \nu$.

Proof. For $\dim X = 0$ the only subspace is $X = 0$. Let $0 < \nu \leq n$. If $X = \text{Span}(u_1, \ldots, u_\nu)$ for a basis $(u_k)_1^n \subset F$ then $\dim X = \nu$. □

Actually,

$$\boxed{\dim(\text{Span}(u_1, \ldots, u_\nu)) = \nu} \tag{1.4}$$

for any linearly independent system $(u_k)_1^\nu$.

COROLLARY 1.2.4. *If X is a subspace such that* $\dim X = n$ *then* $X = F$.

A chain of subspaces

$$X_1 \subset X_2 \subset \ldots \subset X_n \tag{1.5}$$

such that $\dim X_\nu = \nu$, $1 \leq \nu \leq n$, (so that $X_n = F$) is called a **filtration** of the space F. A basis $(u_\nu)_1^n$ is called **compatible** with filtration (1.5) if

$$X_\nu = \text{Span}(u_1, \ldots, u_\nu), \quad 1 \leq \nu \leq n.$$

COROLLARY 1.2.5. *For any filtration there exists a compatible basis.*

Proof. This is trivial for $n = 1$. Omitting X_n in (1.5) we obtain a filtration for X_{n-1}. Let $(u_\nu)_1^{n-1}$ be a compatible basis of X_{n-1}. Since $\dim X_{n-1} < n$, we have $X_{n-1} \neq X_n$ and we can take a vector $u_n \notin X_{n-1}$, i.e. $u_n \notin \mathrm{Span}(u_1, \dots, u_{n-1})$. The system $(u_\nu)_1^n$ is linearly independent so, this is a basis in F. □

THEOREM 1.2.6. *Any basis consists of* n *elements.*

Proof. Suppose that we have two bases, say, $(u_k)_1^n$ and $(v_k)_1^l$. If $l < n$, then by Lemma 1.1.4 $(u_k)_1^n$ are linearly dependent; if $n < l$, then $(v_k)_1^l$ are linearly dependent by the same lemma. Hence $l = n$.□

Theorem 1.2.6 allows us to find the dimensions of many linear spaces.

EXAMPLE 1.2.7 Over $\mathbf{R}$ we have

$$\dim \mathbf{R} = 1, \quad \dim \mathbf{C} = 2, \quad \dim \mathbf{H} = 4. \tag{1.6}$$

□

EXAMPLE 1.2.8. In the matrix space $M_{n,m}(\mathbf{K})$ the matrices $e_{ik} = [\delta_{ij}\delta_{kl}]$, $1 \le i, j \le n$, $1 \le k, l \le m$, form the *canonical* basis. We see that

$$\boxed{\dim M_{n,m}(\mathbf{K}) = nm} \ . \tag{1.7}$$

In particular,

$$\boxed{\dim M_{n,n}(\mathbf{K}) = n^2} \ . \tag{1.8}$$

□

Further we use the following special notation: $M_n(\mathbf{K}) \equiv M_{n,n}(\mathbf{K})$, $\mathbf{K}^n \equiv M_{n,1}(\mathbf{K})$, $(\mathbf{K}^n)' = M_{1,n}(\mathbf{K})$. Thus, $\mathbf{K}^n$ is the set of columns of height n with scalar entries, $(\mathbf{K}^n)'$ is the set of rows of length n. Later on we consider $\mathbf{K}^n$ as a right linear space only, while $(\mathbf{K}^n)'$ will be a left one. Both are called the **arithmetic linear spaces** over $\mathbf{K}$. It follows from (1.7) that

$$\boxed{\dim \mathbf{K}^n = n, \quad \dim (\mathbf{K}^n)' = n} \ . \tag{1.9}$$

It is convenient to consider $M_{n,m}(\mathbf{K})$ as the right linear space of m-tuples whose entries are the columns of height n (in this sense $M_{n,m}(\mathbf{K}) \equiv (\mathbf{K}^n)^m$ and, at the same time, $(M_{n,m}(\mathbf{K}))' \equiv ((\mathbf{K}^m)')^n$ is the left linear space of n-tuples whose entries are the rows of length m).

Note that the left linear structure on $M_{n,m}(\mathbf{K})$ is essentially different from the right one.

EXAMPLE 1.2.9. In the matrix

$$a = \begin{bmatrix} 1 & \beta \\ \alpha & \beta\alpha \end{bmatrix}. \tag{1.10}$$

the columns are right linearly independent if $\alpha\beta \neq \beta\alpha$, otherwise, there exists $\lambda \in \mathbf{K}$ such that $\beta = \lambda$ and $\beta\alpha = \alpha\lambda$, i.e. $\beta\alpha = \alpha\beta$. On the other hand, the columns are left linearly dependent. □

Also an important consequence of Theorem 1.2.6 is the following

COROLLARY 1.2.10. *Any complete system consisting of* n *vectors is a basis.*

Proof. Let $Z = (u_1, \dots, u_n)$ be a complete system. We have to prove that the vectors $u_1, \dots, u_n$ are linearly independent. If not, then $Z \subset \mathrm{Span}(u_{i_1}, \dots, u_{i_l})$, where the vectors $u_{i_1}, \dots, u_{i_l}$ are linearly independent and $l < n$. Since Z is complete, the system $(u_{i_1}, \dots, u_{i_l})$ is also complete by Lemma 1.1.2. Therefore, $u_{i_1}, \dots, u_{i_l}$ is a basis, which contradicts the previous theorem since $l < n$.□

For any system $(v_k)_1^{\nu}$ of vectors in F the dimension of their linear span is called the **rank** of the system. It follows immediately from Lemma 1.1.2 that *the maximal number of linearly independent vectors among* $v_1, \dots, v_\nu$ *coincides with the rank of this system.*

PROPOSITION 1.2.11. *Any system of linearly independent vectors can be extended to a basis.*

Proof. Let $(u_1, \dots, u_\nu)$ be linearly independent system, $\nu = n - k, \quad 0 \le k \le n$. We prove the lemma by induction on k. For $k = 0$ the lemma is true according to Theorem 1.2.1.

Suppose that the lemma is true for some $k \le n-1$ and prove it for $k+1$. Since $\nu = n-(k+1) < n$, the system under consideration is not complete by Theorem 1.2.6. Therefore there exists a vector $u_{\nu+1} \notin \mathrm{Span}(u_1, \dots, u_\nu)$. The system $(u_1, \dots, u_\nu, u_{\nu+1})$ is linearly independent. By induction, this can be extended to a basis. □

COROLLARY 1.2.12. *Any basis of a subspace* $X \subset F$ *can be extended to a basis of the whole* F*.*

The field $\mathbf{K}$ is a linear space over any its subfield $\mathbf{L} \subset \mathbf{K}$. If $\mathbf{K}$ is finite-dimensional over $\mathbf{L}$ then $\mathbf{K}$ is called a *finite extension* of $\mathbf{L}$ and its dimension over $\mathbf{L}$ is called the **degree** of this extension. The standard notation is $[\mathbf{K} : \mathbf{L}]$.

Every linear space F over $\mathbf{K}$ is automatically a linear space over any subfield $\mathbf{L}$. If $\mathbf{L} = \mathbf{R} \subset \mathbf{K}$ then the restriction of the field $\mathbf{K}$ to $\mathbf{L}$ is called the **realification**.

THEOREM 1.2.13.

$$\boxed{\dim_{\mathbf{L}} F = [\mathbf{K} : \mathbf{L}] \cdot \dim_{\mathbf{K}} F} \quad . \tag{1.11}$$

Proof. Let $(v_i)_1^m$ be a basis in F over $\mathbf{K}$ and let $(\lambda_j)_1^p$ be a basis in $\mathbf{K}$ over $\mathbf{L}$. Then the products $v_i \lambda_j$ $(1 \le i \le m, \; 1 \le j \le p)$ form a basis in F over $\mathbf{L}$. Indeed, let $x \in F$. Then we have the decomposition

$$x = \sum_{i=1}^{m} v_i \mu_i$$

with some coefficients $\mu_i \in \mathbf{K}$. In turn,

$$\mu_i = \sum_{j=1}^{p} \lambda_j \alpha_{ij}$$

with some coefficients $\alpha_{ij} \in \mathbf{L}$. Hence,

$$x = \sum_{i=1}^{m} v_i \sum_{j=1}^{p} \lambda_j \alpha_{ij} = \sum_{i,j=1}^{m,p} (v_i \lambda_j) \alpha_{ij}.$$

It remains to show that the system $(v_i\lambda_j)$ is linearly independent. Let

$$\sum_{i,j=1}^{m,p} (v_i\lambda_j)\beta_{ij} = 0$$

with some coefficients β_{ij}. Then

$$\sum_{i=1}^{m} v_i \sum_{j=1}^{p} \lambda_j\beta_{ij} = 0,$$

hence,

$$\sum_{j=1}^{p} \lambda_j\beta_{ij} = 0, \quad 1 \le j \le m,$$

since the system (v_i) is linearly independent. This implies that all $\beta_{ij} = 0$, since the system (λ_j) is also linearly independent.□

EXAMPLE 1.2.14. *If F is a complex linear space then*

$$\boxed{\dim_{\mathbf{R}} F = 2\dim_{\mathbf{C}} F} \tag{1.12}$$

.

EXAMPLE 1.2.15. *If F is a quaternionic linear space then*

$$\boxed{\dim_{\mathbf{C}} F = 2\dim_{\mathbf{H}} F, \quad \dim_{\mathbf{R}} F = 4\dim_{\mathbf{H}} F} \tag{1.13}$$

.

1.3 The coordinates descriptions

THEOREM 1.3.1. *Let (u_i) be a basis. Then each vector z can be uniquely represented as a linear combination of the basis vectors.*

Proof. Since any basis is complete,

$$z = \sum_{i=1}^{n} u_i\zeta_i$$

with some coefficients ζ_i. The coefficients are uniquely determined because of linear independence of the basis vectors. Indeed, if

$$\sum_{i=1}^{n} u_i\zeta_i = \sum_{i=1}^{n} u_i\theta_i.$$

then

$$\sum_{i=1}^{n} u_i(\zeta_i - \theta_i) = 0$$

hence

$$\zeta_i - \theta_i = 0, \quad 1 \le i \le n.$$

□

If (u_i) is a basis and

$$z = \sum_{i=1}^{n} u_i \zeta_i$$

then the coefficients in this decomposition are called the **coordinates** of z for this basis. They are scalar valued functions of $z : \zeta_i = \zeta_i(z)$, $1 \le i \le n$. Obviously,

$$\boxed{\zeta_i(u_k) = \delta_{ik}.} \quad . \tag{1.14}$$

Let us figure out what happens with the coordinates when the basis (u_i) changes for a basis (v_k). To this end we decompose all "new" basis vectors for the "old" ones,

$$v_k = \sum_{k=1}^{n} u_i \tau_{ik}, \quad 1 \le k \le n.$$

The $n \times n$ matrix $t = [\tau_{ik}]$ is called the **matrix of transition** from the basis (u_i) to the basis (v_k). Symbolically,

$$(u_i) \underset{t}{\to} (v_k). \tag{1.15}$$

With any coefficients $\omega_1, \ldots, \omega_n$ we obtain

$$\sum_{k=1}^{n} v_k \omega_k = \sum_{i=1}^{n} u_i \sum_{k=1}^{n} \tau_{ik} \omega_k.$$

Thus, *the "old" coordinates* $\zeta_1, \ldots, \zeta_n$ *of an arbitrary vector* z *are*

$$\boxed{\zeta_i = \sum_{k=1}^{n} \tau_{ik} \omega_k, \quad 1 \le i \le n} \quad , \tag{1.16}$$

where $\omega_1, \ldots, \omega_n$ *are its "new" coordinates.*

Let us assign to a vector x the *column* of its coordinates for each basis under consideration,

$$[\zeta] = \begin{bmatrix} \zeta_1 \\ \cdot \\ \cdot \\ \zeta_n \end{bmatrix} \quad \text{for } (u_i), \quad [\omega] = \begin{bmatrix} \omega_1 \\ \cdot \\ \cdot \\ \omega_n \end{bmatrix} \quad \text{for } (v_k).$$

These columns are actually some $n \times 1$ matrices. In terms of the standard matrix multiplication formula (1.16) can be rewritten as

$$\boxed{[\zeta] = t[\omega]} \quad . \tag{1.17}$$

Conversely, if a matrix $\widetilde{t} \in M_n(\mathbf{K})$ is such that $[\zeta] = \widetilde{t}[\omega]$ for all $x \in F$ then $\widetilde{t} = t$. Indeed, $(t-\widetilde{t})[\omega] = 0$ or equivalently

$$\sum_{k=1}^{n} (\tau_{ik} - \widetilde{\tau}_{ik})\omega_k = 0.$$

Taking $\omega_k = \delta_{ik}$ we obtain $\tau_{ik} = \widetilde{\tau}_{ik}$.

Now let us consider two subsequent transitions, say

$$(u_i) \underset{t}{\rightarrow} (v_k) \underset{s}{\rightarrow} (w_j).$$

Then $[\zeta] = t[\omega]$, $[\omega] = s[\theta]$, where $[\theta]$ is the column of coordinates of x for the third basis (w_j). Since the matrix multiplication is associative, we obtain $[\zeta] = t(s[\theta]) = (ts)[\theta]$. This means that *$ts$ is the transition matrix for the resulting transition $(u_i) \to (w_j)$.*

LEMMA 1.3.2. *Let $t = [\tau_{ik}]$ be the matrix of transition from a basis (u_i) to a basis (v_k). If $(\widetilde{u}_i)$ is also a basis then the system*

$$\widetilde{v}_k = \sum_{i=1}^{n} \widetilde{u}_i \tau_{ik}, \quad 1 \le k \le n, \tag{1.18}$$

is a basis.

Proof. By Theorem 1.2.1 it is sufficient to prove that the system $(\widetilde{v}_k)$ is linearly independent. Let

$$\sum_{k=1}^{n} \widetilde{v}_k \alpha_k = 0$$

with some coefficients α_k. By substitution from (1.18)

$$\sum_{i=1}^{n} \widetilde{u}_i \sum_{k=1}^{n} \tau_{ik} \alpha_k = 0.$$

Since $(\widetilde{u}_i)$ is a basis, we get

$$\sum_{k=1}^{n} \tau_{ik} \alpha_k = 0, \quad 1 \le i \le n.$$

Therefore

$$\sum_{i=1}^{n} u_i \sum_{k=1}^{n} \tau_{ik} \alpha_k = 0$$

hence,

$$\sum_{k=1}^{n} v_k \alpha_k = 0.$$

Since $v_1, \dots, v_n$ are linearly independent, we obtain $\alpha_k = 0$, $1 \le k \le n$. $\square$

1.4 The sums of subspaces

If X and Y are subspaces then their **sum** $X + Y$ is defined as the set of all vectors $x + y$ with $x \in X$ and $y \in Y$. This is a subspace as well. Similarly, the sum with any numbers of summands can be considered.

Let $X_1, \dots, X_l$ be some subspaces. Then their sum is said to be the **direct sum** if for any $x = x_1 + \dots + x_l$ with $x_1 \in X_1, \dots, x_l \in X_l$ the summands are uniquely determined or, equivalently, if $x_1 + \dots + x_l = 0$ with $x_1 \in X_1, \dots, x_l \in X_l$ then $x_1 = \dots = x_l = 0$. In this case the subspaces $X_1, \dots, X_l$ are called **independent** and their direct sum is denoted by $X_1 \dot{+} \dots \dot{+} X_l$. For $l = 2$ the independence is equivalent to $X_1 \cap X_2 = 0$.

THEOREM 1.4.1. *Let $X_1, \dots, X_l$ be independent subspaces and let X be their direct sum. Then for any bases*

$$B_1 = (u_{1i_1})_1^{n_1}, \dots, B_l = (u_{li_l})_1^{n_l}$$

of $X_1, \dots, X_l$ its union B is a basis of X.

Proof. Let $x \in X$, so

$$x = \sum_{k=1}^{l} x_k$$

where $x_k \in X_k$, $1 \le k \le l$. In turn,

$$x_k = \sum_{i=1}^{n_k} u_{ki}\alpha_{ki}.$$

with some coefficients α_{ki} . As a result,

$$x = \sum_{k=1}^{l} \sum_{i=1}^{n_k} u_{ki}\alpha_{ki}.$$

Thus, the system B is complete in X.

In order to prove that B is linearly independent suppose that

$$\sum_{k=1}^{l} \sum_{i=1}^{n_k} u_{ki}\alpha_{ki} = 0$$

with some coefficients α_{ki}. The sum

$$x_k = \sum_{i=1}^{n_k} u_{ki}\alpha_{ki}$$

belongs to X_k, $1 \le k \le l$. Moreover,

$$\sum_{k=1}^{l} x_k = 0.$$

Since the spaces $X_1, \dots, X_l$ are independent, we obtain

$$x_k = 0, \quad 1 \le k \le l.$$

Then $\alpha_{ki} = 0$ since each system B_k is linearly independent. □

COROLLARY 1.4.2. *If the subspaces $X_1, \dots, X_l$ are independent then*

$$\boxed{\dim(X_1 \dot{+} \dots \dot{+} X_l) = \dim X_1 + \dots + \dim X_l} \quad . \tag{1.19}$$

For a subspace $X \subset F$ its **direct complement** Y is defined as a subspace such that$X \dot{+} Y = F$. According to Corollary 1.4.2 we have $\dim X + \dim Y = n$ in this case. Equivalently, $\dim Y = \text{codim} X$ where **codimension** of the subspace X is $\text{codim} X = n - \dim X$.

The subspaces of codimension 1 are called **hyperplanes**.

THEOREM 1.4.3. *For any subspace $X \subset F$ there exists a direct complement.*

Proof. Let $(u_k)_1^\nu$ be a basis of X. By Proposition 1.2.11 this system is contained in a basis $(u_1, \dots, u_\nu, u_{\nu+1}, \dots, u_n)$. Then $Y = \text{Span}(u_{\nu+1}, \dots, u_n)$ is a direct complement of X. □

REMARK 1.4.4. The direct complement is not unique except for the extremal cases $X = 0$, $X = F$, i.e. $\text{codim} X = n$, codimX = 0 respectively. □

1.5 The group $\text{GL}_n(\mathbf{K})$

Let us introduce the set $T_n(\mathbf{K})$ of transition matrices corresponding to all ordered pairs of bases (u_i), (v_k).

THEOREM 1.5.1. *$T_n(\mathbf{K})$ is a group.*

Proof. Let $t, s \in T_n(\mathbf{K})$, $t = [\tau_{ik}]$ and $s = [\sigma_{kj}]$. Take a basis (u_i) and consider the systems

$$v_k = \sum_{i=1}^n u_i \tau_{ik}, \quad 1 \le k \le n,$$

and

$$w_j = \sum_{k=1}^n v_k \sigma_{kj}, \quad 1 \le j \le n.$$

According to Lemma 1.3.2 these systems are bases. Therefore, ts is the matrix of transition $(u_i) \to (w_j)$, i.e. $ts \in T_n(\mathbf{K})$. Thus, $T_n(\mathbf{K})$ is a subsemigroup of the semigroup $M_n(\mathbf{K})$. Note that the **unit matrix** $e = [\delta_{ji}] \in M_n(\mathbf{K})$ is the two-sided unity (neutral element) with respect to the matrix multiplication. The same role of e in $T_n(\mathbf{K})$ can be also seen from the transitions

$$(u_i) \underset{e}{\to} (u_i) \underset{t}{\to} (v_k), \quad (u_i) \underset{t}{\to} (v_k) \underset{e}{\to} (v_k).$$

For the transitions

$$(u_i) \underset{t}{\to} (v_k) \underset{s}{\to} (u_j)$$

the resulting matrix is e hence, $ts = e$. Similarly, $st = e$ follows from

$$(v_k) \underset{s}{\to} (u_j) \underset{t}{\to} (v_k).$$

We see that all matrices from $T_n(\mathbf{K})$ are two-sided invertible. The inverse matrix $s = t^{-1}$ corresponds to the inverse transition. □

COROLLARY 1.5.2. *Every transition matrix is two-sided invertible.*

THEOREM 1.5.3. *If a matrix t is left- or right- invertible then $t \in T_n(\mathbf{K})$.*

Proof. Let $t = [\tau_{ik}]$ be left-invertible, $st = e$. Take a basis (u_i) and consider the system

$$v_k = \sum_{i=1}^{n} u_i \tau_{ik}, \quad 1 \leq k \leq n.$$

We have to prove that (v_k) is a basis. Assume that

$$\sum_{k=1}^{n} v_k \alpha_k = 0$$

with some coefficients α_k. Then

$$\sum_{k=1}^{n} \tau_{ik} \alpha_k = 0$$

i.e. $t[\alpha] = 0$ where $[\alpha]$ is the column of α_k. Multiplying by s on the left hand, we obtain $[\alpha] = 0$. Thus, every left-invertible matrix is a transition matrix.

Now let t be right-invertible, $ts = e$. Then s is left-invertible, so $s \in T_n(\mathbf{K})$ as aforeproved. By Corollary 1.5.2 s is two-sided invertible. Therefore

$$t = te = t(ss^{-1}) = (ts)s^{-1} = es^{-1} = s^{-1}$$

hence, $t \in T_n(\mathbf{K})$. □

COROLLARY 1.5.4. *If a matrix is one-sided invertible then it is two-sided invertible.*

`From now on we can say simply "invertible" in any case.`

COROLLARY 1.5.5. *The set* $\mathrm{GL}_n(\mathbf{K})$ *of all invertible matrices coincides with the group* $T_n(\mathbf{K})$.

Thus, $\mathrm{GL}_n(\mathbf{K})$ *is a group.* In addition we have the following

PROPOSITION 1.5.6. *A $n \times n$ matrix $t = [\tau_{ik}]$ is invertible if and only if its columns (rows) are right (left) linearly independent.*

Proof. Let $t = [\tau_{ik}]$ be invertible. Assume that

$$\sum_{k=1}^{n} \tau_{ik}\alpha_k = 0, \quad 1 \le i \le n,$$

and

$$\sum_{i=1}^{n} \beta_i \tau_{ik} = 0, \quad 1 \le k \le n,$$

with some coefficients α_k, β_i. In the matrix notation $t[\alpha] = 0$ and $[\beta]'t = 0$ where $[\beta]'$ is the transposed column $[\beta]$. Multiplying these equations by t^{-1} on the left and right respectively we obtain $[\alpha] = 0$, $[\beta] = 0$.

Now let the columns of $t = [\tau_{ik}]$ are right linearly independent. We prove that t is a transition matrix. Then t is invertible by Corollary 1.5.2.

Take a basis (v_i) and

$$u_k = \sum_{i=1}^{n} v_i \tau_{ik}, \quad 1 \le k \le n.$$

We need to prove that (u_k) is linearly independent.

Assume that

$$\sum_{k=1}^{n} u_k \alpha_k = 0$$

with some coefficients α_k. Then

$$\sum_{k=1}^{n} \tau_{ik} \alpha_k = 0$$

since (v_i) are linearly independent. Since the columns of t are right linearly independent, we get $[\alpha] = 0$.

Similarly, if the rows are left linearly independent then t is invertible. □

EXAMPLE 1.5.7.Consider the matrix a defined by (1.10) with $\alpha\beta \ne \beta\alpha$, so that the columns are right linearly independent. Thus, a is invertible by Proposition 1.5.6. Nevertheless, the columns of a are left linearly dependent.

Let us find the inverse matrix

$$a^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

Then the equality $aa^{-1} = e$ is equivalent to the following system

$$\begin{cases} \alpha_{11} & + & \beta\alpha_{21} & = & 1 \\ \alpha_{12} & + & \beta\alpha_{22} & = & 0 \\ \alpha\alpha_{11} & + & \beta\alpha\alpha_{21} & = & 0 \\ \alpha\alpha_{12} & + & \beta\alpha\alpha_{22} & = & 1. \end{cases}$$

From the first equality we get

$$\alpha_{11} = 1 - \beta\alpha_{21}.$$

Substituting it in the third equality we obtain

$$\alpha = -\Delta\alpha_{21}$$

where

$$\Delta = \beta\alpha - \alpha\beta \neq 0.$$

So,

$$\alpha_{21} = -\Delta^{-1}\alpha, \quad \alpha_{11} = 1 + \beta\Delta^{-1}\alpha.$$

Using the second equality we get

$$\alpha_{12} = -\beta\alpha_{22}.$$

Substituting this in the last equality we obtain

$$\alpha_{22} = \Delta^{-1}.$$

Thus,

$$\alpha_{12} = -\beta\Delta^{-1}.$$

Finally,

$$a^{-1} = \begin{bmatrix} 1 + \beta\Delta^{-1}\alpha & -\beta\Delta^{-1} \\ -\Delta^{-1}\alpha & \Delta^{-1} \end{bmatrix}.$$

□

1.6 Linear functionals

A mapping $\varphi : F \to \mathbf{K}$ (i.e. a scalar valued function $\varphi(x)$, $x \in F$) is called a **linear functional** on F if it is **additive**, i.e.

1) $\varphi(x+y) = \varphi x + \varphi y \quad (x, y \in F)$

and **homogeneous,**

2) $\varphi(x\alpha) = (\varphi(x))\alpha \quad (x \in F, \quad \alpha \in \mathbf{K})$.

The set of all linear functionals on F is denoted by F'.

We define the **sum** of linear functionals $\varphi_1, \varphi_2 \in F'$ as

$$(\varphi_1 + \varphi_2)(x) = \varphi_1(x) + \varphi_2(x).$$

It is easy to see that $\varphi_1 + \varphi_2 \in F'$ and F' is an Abelian group provided with this addition.

If $\varphi \in F'$ and $\lambda \in \mathbf{K}$ we define $\lambda\varphi$ as

$$(\lambda\varphi)(x) = \lambda(\varphi(x))$$

where the right hand side is the product in $\mathbf{K}$. Then $\lambda\varphi \in F$. We only check the homogeneity:

$$(\lambda\varphi)(x\alpha) = \lambda(\varphi(x\alpha)) = \lambda(\varphi(x)\alpha) = (\lambda(\varphi(x)))\alpha = ((\lambda\varphi)(x))\alpha.$$

Thus, with the above introduced operations *the set F' is a left linear space* over the same field $\mathbf{K}$. This space is called **dual** to F.

PROPOSITION 1.6.1. *Let $(u_k)_1^n$ be a basis of F. Then the coordinates $\xi_1(x), \dots, \xi_n(x)$ are linear functionals of $x \in F$.*

Proof. Let $x, y \in F$. Then we have the following decompositions

$$x = \sum_{k=1}^{n} u_k \xi_k(x), \qquad y = \sum_{k=1}^{n} u_k \xi_k(y).$$

Hence,

$$x + y = \sum_{k=1}^{n} u_k(\xi_k(x) + \xi_k(y))$$

which means that

$$\xi_k(x + y) = \xi_k(x) + \xi_k(y), \quad 1 \le k \le n.$$

Similarly,

$$x\alpha = \sum_{k=1}^{n} u_k \xi_k(x\alpha) = \sum_{k=1}^{n} u_k \xi_k(x)\alpha.$$

Hence,

$$\xi_k(x\alpha) = \xi_k(x)\alpha.$$

□

PROPOSITION 1.6.2. *The system $(\xi_k)_1^n$ is a basis of F'.*

This basis is called **dual** to $(u_k)_1^n$ in view of relations

$$\boxed{\xi_i(u_k) = \delta_{ik}} \quad . \tag{1.20}$$

In view of (1.20) the basis $(\xi_k)_1^n$ is also called **biorthogonal** to $(e_k)_1^n$.

Proof. Let

$$\sum_{k=1}^{n} \gamma_k \xi_k = 0$$

for some coefficients $\gamma_1, \dots, \gamma_n$. Then

$$0 = \left(\sum_{k=1}^{n} \gamma_k \xi_k \right)(u_i) = \sum_{k=1}^{n} \gamma_k \xi_k(u_i) = \sum_{k=1}^{n} \gamma_k \delta_{ki} = \gamma_i, \quad 1 \le i \le n,$$

i.e. the system $(\xi_k)_1^n$ is linearly independent.

Let now φ be an arbitrary linear functional on F and

$$x = \sum_{k=1}^{n} u_k \xi_k(x).$$

Then

$$\varphi(x) = \varphi\Big(\sum_{k=1}^n u_k \xi_k(x)\Big) = \sum_{k=1}^n \varphi(u_k)\xi_k(x) \tag{1.21}$$

for all x. Hence

$$\varphi = \sum_{k=1}^n \varphi(u_k)\xi_k,$$

i.e., the system $(\xi_k)_1^n$ is complete. □

COROLLARY 1.6.3. $\dim F' = n$, *i.e.* $\dim F' = \dim F$.

We also see from above that *the coordinates* ε_k *of the linear functional* φ *for the dual basis in* F' *are the values* $\varphi(u_k)$ *on the given basis in* F. We assign to the linear functional φ the *row*

$$[\varepsilon] = [\varepsilon_1, \ldots, \varepsilon_n]$$

of its coordinates for the dual basis. Then

$$\varphi(x) = [\varepsilon][\xi]$$

where $[\xi]$ is the coordinate column of x for the basis $(u_k)_1^n$.

In this context it is convenient to use the term **covectors** for the linear functionals.

If t is the matrix of a transition to a new basis then $\varphi(x) = [\varepsilon]t[\eta]$ where $[\eta]$ is the coordinate column of x for the new basis. We see that **the coordinate row** of the covector φ for the new dual basis is

$$\boxed{[\delta] = [\varepsilon]t} \quad . \tag{1.22}$$

Thus, in the dual space the transition matrix acts on the right hand side in contrast to (1.17).

In conclusion we prove the following useful

PROPOSITION 1.6.4. *A subspace* $X \subset F$ *coincides with whole* F *if and only if the only linear functional* φ *such that* $\varphi|X = 0$ *is* $\varphi = 0$.

Proof. Let $X \neq F$ so, $\dim X = m < n$. Consider a basis $(u_k)_1^n \subset F$ such that $(u_k)_1^m$ is a basis in X (see Corollary 1.2.12). Then the coordinate ξ_{m+1} is a nonzero linear functional such that $\xi_{m+1}|X = 0$. □

COROLLARY 1.6.5. *A system of vectors* $(v_k)_1^l$ *is complete in* F *if and only if the only linear functional* φ *such that*

$$\varphi(v_k) = 0, \quad 1 \leq k \leq l, \tag{1.23}$$

is $\varphi = 0$.

Proof. The system of equations (1.23) is equivalent to $\varphi|X = 0$ where $X = \mathrm{Span}(v_1, \ldots, v_l)$. □

1.7 Homomorphisms and linear operators

Consider one more right linear space E over the same field $\mathbf{K}$, $\dim E = m$.

A mapping $f : E \to F$ is called a *homomorphism* from E into F if it is **linear**, i.e. additive

1)$f(x+y) = f\,x + f\,y \quad (x, y \in E)$

and homogeneous,

2)$f(x\alpha) = (f\,x)\alpha \quad (x \in E, \quad \alpha \in \mathbf{K})$,

like linear functionals. So, the linear functionals are just homomorphisms $E \to \mathbf{K}$.

The set of all homomorphisms from E into F is denoted by $\mathrm{Hom}(E, F)$. In particular, $\mathrm{Hom}(E, \mathbf{K}) \equiv E'$. In the case $E = F$ the homomorphisms are called **endomorphisms** of E or **linear operators** in E. The corresponding notation is $L(E) \equiv \mathrm{End}(E) \equiv \mathrm{Hom}(E, E)$.

The space $\mathrm{Hom}(E, F)$ provided with the standard addition is an Abelian group. However, there is no natural multiplication by scalars in $\mathrm{Hom}(E, F)$ since the definition

$$(\lambda f)x = \lambda(fx)$$

makes no sense if the right linear space F is not left one. If F is a two-sided linear space then $\mathrm{Hom}(E, F)$ can be provided with a structure of left-linear space like E'.

Let f be a homomorphism $E \to F$. It is called an **epimorphism** if it is a surjective mapping, i.e. for any $y \in F$ there exists $x \in E$ such that $y = fx$. In other words, f is an epimorphism if and only if its **image**,

$$\mathrm{Im} f = \{y \in F : y = fx\},$$

is the whole F. Note that the image of any $f \in \mathrm{Hom}(E, F)$ s a subspace of the space E.

A homomorphism $f : E \to F$ is called a **monomorphism** if it is an injective mapping, i.e. equality $fx_1 = fx_2$ implies $x_1 = x_2$. This property can be characterized in terms of the **kernel**,

$$\mathrm{Ker} f = \{x \in E : fx = 0\}.$$

Namely, f is a monomorphism if and only if $\mathrm{Ker} f = 0$. Indeed, if $\mathrm{Ker} f = 0$ then

$$fx_1 = fx_2 \Rightarrow f(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in \mathrm{Ker} f \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

Conversely, if f is a monomorphism and $x \in \mathrm{Ker} f$ then $fx = 0$ and $f0 = 0$. Hence, $x = 0$.

Any bijective homomorphism $f : E \to F$ is called an **isomorphism** from E onto F. Obviously, $f \in \mathrm{Hom}(E, F)$ is an isomorphism if and only if f is an epimorphism and a monomorphism at the same time, i.e.

$$\mathrm{Ker} f = 0, \quad \mathrm{Im} f = F$$

simultaneously.

Having an isomorphism $f : E \to F$ one can identify F with E ($E \equiv F$) by identification $fx \equiv x$ for all $x \in E$.

Any monomorphism $f : E \to F$ defines the isomorphism $\widetilde{f} : E \to \mathrm{Im} f$ which allows us to identify $\mathrm{Im} f$ with E. On the other hand, for any subspace $X \subset F$ the mapping $i : X \to F$ defined as

$ix = x, \quad x \in X$, is a monomorphism trivially. This monomorphism is called the **embedding** of the subspace X into the whole space F. Therefore any monomorphism $f : E \to F$ can be also called an **embedding** of E into F by identification $E \equiv \mathrm{Im} f$.

For any $f \in \mathrm{Hom}(E, F)$ one can introduce two integer numbers,

$$\mathrm{rank}(f) = \dim(\mathrm{Imf}), \quad \mathrm{def}(f) = \dim(\mathrm{Ker} f),$$

the **rank** and the **defect** of f. Obviously,

$$0 \le \mathrm{rank}(f) \le n, \quad 0 \le \mathrm{def}(f) \le m$$

and

$$\mathrm{rank}(f) = 0 \Leftrightarrow \mathrm{def}(f) = m \Leftrightarrow f = 0.$$

THEOREM 1.7.1. *For any* $f \in \mathrm{Hom}(E, F)$ *the relation*

$$\boxed{\mathrm{rank}(f) + \mathrm{def}(f) = \dim E.} \tag{1.24}$$

is valid.

Proof. Let $\mathrm{def}(f) = l$ and let $(e_k)_1^l$ be a basis of $\mathrm{Ker} f$. According to Proposition 1.2.11 this basis can be extended to a basis $(e_k)_1^m$ of the whole E. It is sufficient to prove that $(f(e_k))_{l+1}^m$ is a basis in $\mathrm{Im} f$.

Any vector from $\mathrm{Im} f$ is of form

$$f(\sum_{k=1}^m e_k \alpha_k) = \sum_{k=1}^m f(e_k)\alpha_k = \sum_{k=l+1}^m f(e_k)\alpha_k$$

since $f(e_k) = 0, \quad 1 \le k \le l$. Thus, $\mathrm{Im} f = \mathrm{Span}(f(e_{l+1}), \ldots, f(e_m))$. It remains to prove that the system $(f(e_k))_{l+1}^m$ is linearly independent.

Suppose that

$$\sum_{k=l+1}^m f(e_k)\gamma_k = 0.$$

Then

$$f(\sum_{k=l+1}^m e_k\gamma_k) = 0,$$

i.e.

$$\sum_{k=l+1}^m e_k\gamma_k \in \mathrm{Ker} f.$$

The latter vector can be decomposed for the basis $(e_k)_1^l \in \mathrm{Ker} f$:

$$\sum_{k=l+1}^m e_k\gamma_k = \sum_{k=1}^l e_k\gamma_k.$$

Since $(e_i)_1^m$ is a basis, all coefficients $\gamma_k, \ 1 \le k \le m$, are equal to zero.□

COROLLARY 1.7.2. *The inequality*

$$\boxed{\mathrm{rank}(f) \le \min(m, n)} \qquad (1.25)$$

holds.

The following consequence of Theorem 1.7.1 is especially important.

COROLLARY 1.7.3. *In the case* $m = n$ *(i.e.* $\dim E = \dim F$*) the following properties are equivalent for* $f \in \mathrm{Hom}(E, F)$*:*

a) f *is an epimorphism.*

b) f *is a monomorphism.*

Thus, each property a) *or* b) *implies that* f *is an isomorphism.*

Applying Theorem 1.7.1 to the case $F = \mathbf{K}$ we obtain

COROLLARY 1.7.4. *For any linear functional* $\varphi \neq 0$ *on the space* E

$$\dim(\mathrm{Ker}\varphi) = m - 1.$$

Thus, $\mathrm{Ker}\varphi$ is a hyperplane.
Conversely, we have

PROPOSITION 1.7.5. *For any hyperplane* X *there exists a linear functional* φ *such that* $X = \mathrm{Ker}\varphi$*. All such linear functionals are proportional.*

Proof. If $(e_k)_1^{m-1}$ is a basis in X and $(e_k)_1^m$ is its extension to a basis in E then $X = \mathrm{Ker}\xi_m$ where $\xi_m = \xi_m(x)$ is the corresponding coordinate of vector x.

Now let $\mathrm{Ker}\varphi_1 = \mathrm{Ker}\varphi_2 \neq 0$. Consider a vector v such that $\varphi_1(v) \neq 0$. For any x the vector

$$y = x - v\Big(\varphi_1(v)^{-1}\varphi_1(x)\Big)$$

belongs to $\mathrm{Ker}\varphi_1$. Hence, $\varphi_2(y) = 0$, i.e. $\varphi_2(x) = \lambda\varphi_1(x)$ where $\lambda = \varphi_2(v)\varphi_1(v)^{-1}$. This means that $\varphi_2 = \lambda\varphi_1$. □

The class of right linear spaces over $\mathbf{K}$ is a **category** with homomorphisms as morphisms. Indeed, first of all, we have

PROPOSITION 1.7.6. *Let* E, F, G *be some right linear spaces. If a mapping* $f : E \to F$ *and a mapping* $g : F \to G$ *are homomorphisms then their product (composition)* $gf : E \to G$ *is a homomorphism as well.*

Proof. Obviously,

$$(gf)(x+y) = g(f(x+y)) = g(f\,x + f\,y) = (gf)x + (gf)y$$

and

$$(gf)(x\alpha) = g(f(x\alpha)) = g((fx)\alpha) = ((gf)x)\alpha.$$

□

In addition, the identity homomorphism $\mathrm{id}_F : F \to F$ is a neutral element in the sense $\mathrm{id}_F \cdot f = f$ for all $f \in \mathrm{Hom}(E, F)$ and $g \cdot \mathrm{id}_F = g$ for all $g \in \mathrm{Hom}(F, G)$.

Obviously, id_F is an isomorphism $F \to F$. Any isomorphism $F \to F$ is called an **automorphism** of the space F. The set of all automorphisms $E \to E$ is denoted by $\mathrm{Aut}(E)$.

COROLLARY 1.7.7. *The product of epimorphisms (monomorphisms, isomorphisms) is an epimorphism (monomorphism, isomorphism).*

COROLLARY 1.7.8. *The set $L(E)$ of linear operators in E is a multiplicative monoid.*

The invertible elements of the monoid $L(E)$ form a group $L^{\#}(E)$, the group of invertible linear operators. We prove the following

PROPOSITION 1.7.9. *A linear operator f in E is a bijective mapping $E \to E$ if and only if it is an automorphism.*

Proof. There exists an inverse mapping $f^{-1} : E \to E$,

$$ff^{-1} = f^{-1}f = \mathrm{id}_E,$$

f being bijective. We have to prove that f^{-1} is linear as well. If $x, y \in F$ and

$$f^{-1}(x+y) = z, \quad f^{-1}x = u, \quad f^{-1}y = v$$

then

$$f(u+v) = fu + fv = x + y = fz.$$

Hence,

$$u + v = z$$

so,

$$f^{-1}(x+y) = f^{-1}x + f^{-1}y.$$

We also have

$$f^{-1}(x\alpha) = u\alpha = (f^{-1}x)\alpha,$$

since

$$f(u\alpha) = (fu)\alpha = x\alpha.$$

□

COROLLARY 1.7.10. $L^{\#}(E) = \mathrm{Aut}(E)$, *thus,* $\mathrm{Aut}(E)$ *is a group.*

The same proof as in Proposition 1.7.9 yields a more general

PROPOSITION 1.7.11. *If f is an isomorphism from E onto F then f^{-1} is an isomorphism from F onto E.*

Note that the invertibility of a linear operator is provided by its one-sided invertibility.

PROPOSITION 1.7.12. *If a linear operator f in E is one-sided invertible then it is invertible. Moreover, the only one-sided inverse to f is f^{-1}.*

Proof. If f is right-invertible, i.e. $fg = \mathrm{id}$, then for any $x \in E$ we have $x = f(gx)$. This means that $x \in \mathrm{Im} f$. Let now f is left-invertible, i.e. $gf = \mathrm{id}$. Then for x such that $fx = 0$ we get $x = g(fx) = 0$, i.e. $\mathrm{Ker} f = 0$. Now Corollary 1.7.3 yields that f is invertible in both cases.

Let now f_L^{-1} and f_R^{-1} are some left- and right-inverse to f. Then

$$f_R^{-1} = \mathrm{id} \cdot f_R^{-1} = (f_L^{-1} f) f_R^{-1} = f_L^{-1}(f f_R^{-1}) = f_L^{-1} \cdot \mathrm{id} = f_L^{-1}.$$

Therefore both operators f_R^{-1} and f_L^{-1} are two-sided inverse and the inverse is unique. Thus, $f_R^{-1} = f_L^{-1} = f^{-1}$. □

The space F is called **isomorphic** to E if there exists an isomorphism $E \to F$. This property is denoted by $E \approx F$. This is a binary relation on the category of all right linear spaces. Actually, this is an equivalence, i.e. that it is **reflexive**,

$$E \approx E,$$

since there exists id_E; it is **symmetric**,

$$(E \approx F) \Rightarrow (F \approx E),$$

by Corollary 1.7.11; it is **transitive**

$$(E \approx F) \& (F \approx G) \Rightarrow (E \approx G),$$

by Proposition 1.7.6. Thus, one can speak about the **classes** of isomorphic right spaces.

Obviously, if $E \approx F$ then $\dim E = \dim F$. The converse is also true.

THEOREM 1.7.13. *If $\dim E = m$ then E is isomorphic to the arithmetic m-dimensional space $\mathbf{K}^m$.*

Proof. Given a basis in E. The mapping $x \mapsto [\xi]$, where $[\xi]$ is the coordinate column of x, is an isomorphism $E \to \mathbf{K}^m$. □

Note that the isomorphism we use above maps the given basis in E onto the canonical basis in $\mathbf{K}^m$.

By Theorem 1.7.13 the space $\mathbf{K}^m$ of columns of height m can be chosen as a **canonical representative** of the class of m-dimensional right linear spaces. For the left linear spaces the situation is similar but it is convenient to choose the space $(\mathbf{K}^m)'$ of rows of length m as the canonical representative in this case.

COROLLARY 1.7.14. *If* $\dim E = \dim F$ *then* $E \approx F$.

Proof. Both spaces E and F are m-dimensional. Hence, $E \approx \mathbf{K}^m$ and $F \approx \mathbf{K}^n$. By symmetry and transitivity $E \approx F$. □

Note that the isomorphism $E \to F$ in the proof depends on choice of some bases in E and F.

Now for any vector $u \in F$ and any covector $\varphi \in E'$ one can introduce the **tensor product** $u \otimes \varphi \in \mathrm{Hom}(E, F)$ as

$$(u \otimes \varphi)(x) = u\varphi(x).$$

This is an important construction, but in the proof of the next lemma we use that to only avoid a more complicated notation.

LEMMA 1.7.15. *Let* $(e_k)_1^m$ *be a basis of* E. *Then for a given system* $(z_k)_1^n) \in F$ *there exists a unique homomorphism* $f \in \mathrm{Hom}(E, F)$ *such that*

$$fe_k = z_k, \quad 1 \le k \le m. \tag{1.26}$$

Proof. Let $(\xi_k)_1^m$ be the dual basis in E'. Define homomorphism $f \in \mathrm{Hom}(E, F)$ by

$$f = \sum_{i=1}^{m} z_i \otimes \xi_i.$$

Then

$$fe_k = \sum_{i=1}^{m} z_i \otimes \xi_i(e_k) = z_k.$$

Under conditions (1.26) a homomorphism $f \in \mathrm{Hom}(E, F)$ is unique since

$$fx = f(\sum_{k=1}^{m} e_k \xi_k(x)) = \sum_{k=1}^{m} z_k \xi_k(x), \quad x \in E.$$

□

Now we choose a basis $(u_i)_1^n$ in F and consider the decompositions

$$fe_k = \sum_{i=1}^{n} u_i \alpha_{ik}, \quad 1 \le k \le m, \tag{1.27}$$

so that the coordinates of fe_k are the elements of the k-th column of the matrix $[\alpha_{ik}] \in M_{n,m}(\mathbf{K})$. This matrix is called the **matrix of homomorphism** f for the given bases in $(e_k)_1^m \subset E$ and $(u_i)_1^n \subset F$. It is denoted by $\mathrm{mat}(f)$. In the case $E = F$ we assume that the bases in E and F are the same. In this case $\mathrm{mat}(f)$ runs over square matrices of order m.

EXAMPLE 1.7.16. For any $\lambda \in \mathbf{K}$ consider the scalar matrix $\alpha_{ik} = \lambda\delta_{ik}, \ \ 1 \le i, k \le m$. For the corresponding linear operator $f_\lambda : E \to E$ we have

$$f_\lambda e_k = \sum_{i=1}^{m} e_i(\lambda\delta_{ik}) = e_k\lambda.$$

Therefore for any vector

$$x = \sum_{k=1}^{m} e_k \xi_k$$

its image is

$$f_\lambda x = \sum_{k=1}^{m} (f_\lambda e_k)\xi_k = \sum_{k=1}^{m} (e_k \lambda)\xi_k = \sum_{k=1}^{m} e_k(\lambda \xi_k).$$

We see that f_λ operates in coordinates by left multiplication by λ of all ones. If $\mathbf{K}$ is not commutative then $f_\lambda x \neq \lambda x$ in general. In other words, the operator f_λ is not scalar!

On the other hand, one can consider the mapping $x \mapsto x\lambda$, $x \in E$. *This mapping is not linear if* $\mathbf{K}$ *is not commutative.* Indeed, if $\mu \in \mathbf{K}$, $\mu\lambda \neq \lambda\mu$ then $e_1 \mapsto e_1\lambda$ but

$$e_1\mu \mapsto (e_1\mu)\lambda = e_1(\mu\lambda) \neq e_1(\lambda\mu) = (e_1\lambda)\mu.$$

□

According to Lemma 1.7.15 the mapping $f \mapsto \mathrm{mat}(f)$ is bijective. This mapping preserves all algebraic operations existing in $\mathrm{Hom}(E, F)$, namely:

1) $\mathrm{mat}(g + f) = \mathrm{mat}(g) + \mathrm{mat}(f)$,

i.e. 'mat' is a homomorphism of additive groups (in particular, $\mathrm{mat}(0) = 0$, $\mathrm{mat}(-f) = -\mathrm{mat}(f)$);

2) $\mathrm{mat}(gf) = \mathrm{mat}(g) \cdot \mathrm{mat}(f)$

(under assumption that the matrices are assigned to the relevant bases).

Moreover, $\mathrm{mat}(\mathrm{id}_E)$ is the unit matrix $e = [\delta_{ik}]$, $1 \leq i, k \leq m$. We also have

PROPOSITION 1.7.17. *A linear operator $f \in L(E)$ is invertible if and only if* $\mathrm{mat}(f)$ *is invertible and*

$$\mathrm{mat}(f^{-1}) = (\mathrm{mat}(f))^{-1} \tag{1.28}$$

in this case.

Proof. If f is invertible, $ff^{-1} = \mathrm{id}_E$, then

$$(\mathrm{mat}(f)) \cdot (\mathrm{mat}(f^{-1})) = \mathrm{mat}(ff^{-1}) = \mathrm{mat}(\mathrm{id}_E) = e,$$

hence, $\mathrm{mat}(f)$ is invertible and (1.28) takes place.

Conversely, one can consider the linear operator g such that $\mathrm{mat}(g) = a^{-1}$ where $a = \mathrm{mat}(f)$. Then $\mathrm{mat}(gf) = a^{-1}a = e = \mathrm{mat}(\mathrm{id}_E)$ hence, $gf = \mathrm{id}_E$. Similarly, $fg = \mathrm{id}_E$. □

COROLLARY 1.7.18. *The group* $\mathrm{Aut}(E)$ *is isomorphic to* $\mathrm{GL}_n(\mathbf{K})$.

The 1-1 correspondence $f \mapsto \mathrm{mat}(f)$ between linear operators in E and $m \times m$ matrices appears as soon as a basis in E is chosen. In order to investigate how $\mathrm{mat}(f)$ depends on basis we find out how f effects on the coordinates.

Let $x \in E$

$$x = \sum_{k=1}^{m} e_k \xi_k.$$

It follows from (1.27) that

$$fx = \sum_{i=1}^{n} u_i \sum_{k=1}^{m} \alpha_{ik}\xi_k.$$

Thus, *the coordinates of fx for the basis $(u_i)_1^n$ in F are*

$$\boxed{\eta_i = \sum_{k=1}^{n} \alpha_{ik}\xi_k} \tag{1.29}$$

where ξ_k are coordinates of x for the basis $(e_k)_1^m$ in E. In the matrix form

$$\boxed{[\eta] = \mathrm{mat}(f)[\xi]} \tag{1.30}$$

where $[\xi]$ is the column of coordinates of x for the basis $(e_k)_1^m$ and $[\eta]$ is the column of coordinates of fx for the basis $(u_i)_1^n$.

PROPOSITION 1.7.19. *Let a and b be the matrices of $f \in L(E)$ for some bases $(e_i)_1^m$ and $(v_k)_1^m$ respectively. Then*

$$\boxed{b = t^{-1}at} \tag{1.31}$$

where t is the matrix of transition $(e_i)_1^m \to (v_k)_1^m$.

In this sense b is **similar** to a. If a and b are invertible then they are **conjugate** in $\mathrm{GL}_n(\mathbf{K})$.

Proof. Let $[\zeta]$ and $[\omega]$ be coordinate columns of $x \in E$ for the bases $(e_i)_1^m$ and $(v_k)_1^m$ respectively. Similarly, we have the columns $[\alpha]$ and $[\beta]$ for fx. It follows from (1.17) that

$$[\zeta] = t[\omega], \quad [\alpha] = t[\beta]. \tag{1.32}$$

In turn, (1.30) yields

$$[\alpha] = a[\zeta], \quad [\beta] = b[\omega]. \tag{1.33}$$

Combining (1.32) and (1.33) we obtain

$$at[\omega] = tb[\omega]$$

for all $x = [\omega]$. Thus,

$$at = tb$$

which yields (1.31). □

1.8 The duality theory

We know that $\dim E' = \dim E$ but the sentence "E' is isomorphic to E" makes no sense. Indeed, E is a right linear space while E' is a left one. However, the second dual space $E'' \equiv (E')'$ is a right linear space again and $\dim E'' = \dim E' = \dim E$, hence, $E \approx E''$.

In order to get an isomorphism $E \to E''$ it is sufficient to choose a basis in E and map it onto a basis in E''. However, if the latter is taken as the second dual to the former then the isomorphism is actually independent of the choice of the initial basis. This isomorphism $E \to E''$ is called **canonical**. The canonical image of $x \in E$ will be denoted by $\hat{x}$, $\hat{x} \in E''$.

THEOREM 1.8.1. *The canonical isomorphism $E \to E''$ is given by the formula*

$$\hat{x}(\varphi) = \varphi(x), \quad \varphi \in E'. \tag{1.34}$$

Proof. By definition of the mapping $x \mapsto \hat{x}$ we have

$$\hat{x} = \sum_{k=1}^{m} e''_k e'_k(x)$$

for

$$x = \sum_{k=1}^{m} e_k e'_k(x).$$

Therefore

$$\hat{x}(\varphi) = (\sum_{k=1}^{m} e''_k e'_k(x))(\varphi) = \sum_{k=1}^{m} e''_k(\varphi) e'_k(x).$$

By definition, $e''_k(\varphi)$ are coordinates of φ for the basis $(e'_k)_1^m$. As we know they are $\varphi(e_k)$, $1 \le k \le m$. Hence,

$$\hat{x}(\varphi) = \sum_{k=1}^{m} \varphi(e_k) e'_k(x) = \varphi(x)$$

(see (1.21)). □

From now on we identify E'' with E by the canonical isomorphism.

Let $f \in \mathrm{Hom}(E, F)$. The **adjoint** mapping $f' : F' \to E'$ is defined as such that

$$(f'\varphi)(x) = \varphi(fx) \tag{1.35}$$

for all $\varphi \in F'$ and for all $x \in F$. In other words, $f'\varphi = \varphi f$, the product of $\varphi \in \mathrm{Hom}(F, \mathbf{K})$ and $f \in \mathrm{Hom}(E, F)$.

PROPOSITION 1.8.2. *The adjoint mapping f' is a homomorphism, i.e. $f' \in \mathrm{Hom}(F', E')$.*

Proof. For $x, y \in E$ and $\lambda \in \mathbf{K}$ we obtain

$$(f')\varphi(x + y) = \varphi(f(x + y)) = \varphi(fx) + \varphi(fy) = (f'\varphi)(x) + (f'\varphi)(y)$$

and

$$(f'(\lambda\varphi))(x) = (\lambda\varphi)(fx) = \lambda(\varphi(fx)) = \lambda(f'\varphi)(x) = \big(\lambda(f'\varphi)\big)\,(x).$$

□

Now let us prove some standard properties of the mapping $f \mapsto f'$ from $\mathrm{Hom}(E, F)$ into $\mathrm{Hom}(F', E')$.

First of all, we consider $f'' : E'' \to F''$ as a homomorphism $E \to F$ by canonical identifications $E'' = E, \quad F'' = F$. It turns that

$$\boxed{f'' = f} \quad . \tag{1.36}$$

Indeed, let $\hat{x} \in E''$. Using (1.34) and (1.35) for $\psi \in F'$ we get

$$(f''\hat{x})(\psi) = \hat{x}(f'\psi) = (f'\psi)(x) = \psi(fx) = \widehat{fx}(\psi).$$

Hence, $f''\hat{x} = \widehat{fx}$. By identification $\hat{x} \equiv x$, $\widehat{fx} \equiv fx$ we obtain $f''x = fx$ which means (1.36).□

For $f, g \in \mathrm{Hom}(E, F)$ we have

$$\begin{aligned} (f+g)'\varphi(x) &= \varphi\,((f+g)x) = \varphi(fx+gx) = \varphi(fx) + \varphi(gx) \\ &= (f'\varphi)(x) + (g'\varphi)(x) = (f'\varphi + g'\varphi)(x) = \big((f'+g')\varphi\big)\,(x) \end{aligned}$$

i.e.

$$\boxed{(f+g)' = f' + g'} \quad . \tag{1.37}$$

Now let $f \in \mathrm{Hom}(E, F)$ and $g \in \mathrm{Hom}(F, G)$. Then

$$(f'g')\varphi = f'(g'\varphi) = f'(\varphi g) = (\varphi g)f = \varphi(gf) = (gf)'\varphi,$$

hence,

$$\boxed{(gf)' = f'g'} \quad . \tag{1.38}$$

Note also that

$$\boxed{(\mathrm{id}_F)' = \mathrm{id}_F} \quad , \tag{1.39}$$

since

$$(\mathrm{id}_F)'\varphi = \varphi(\mathrm{id}_F) = \varphi.$$

Also we have another important relation.

PROPOSITION 1.8.3. *Let f be a linear operator in E. Then the adjoint linear operator f' in E' is invertible if and only if f is invertible. In this situation*

$$\boxed{(f')^{-1} = (f^{-1})'} \quad . \tag{1.40}$$

Proof. If f is invertible then

$$\mathrm{id}_{E'} = (\mathrm{id}_E)' = (f^{-1}f)' = f'(f^{-1})'.$$

We see that f' is right-invertible and $(f^{-1})'$ is its right-inverse operator. By Proposition 1.7.12 f' is invertible and (1.40) holds.

Conversely, if f' is invertible, then f'' is invertible. Hence, $f \equiv f''$ is invertible. □

Now let $(e_k)_1^m$ and $(u_i)_1^n$ be some bases in E and F respectively and let $\mathrm{mat}(f) = [\alpha_{ik}]$, i.e.

$$fe_k = \sum_{i=1}^{n} u_i\alpha_{ik}, \quad 1 \le k \le m.$$

In order to introduce the matrix of f' for the dual bases $(\xi_k)_1^m \in E'$ and $(\eta_i)_1^n \in F'$ we observe that

$$(f'\eta_i)(e_k) = \eta_i(fe_k) = \eta_i(\sum_{j=1}^{n} u_j\alpha_{jk}) = \sum_{j=1}^{n} \eta_i(u_j)\alpha_{jk} = \sum_{j=1}^{n} \delta_{ij}\alpha_{jk} = \alpha_{ik}.$$

For this reason we can agree to set

$$\boxed{\mathrm{mat}(f') = \mathrm{mat}(f)}\ , \tag{1.41}$$

however, in the matrix $\mathrm{mat}(f')$ the coordinates of covectors $f'\eta_i \in F'$, $1 \le i \le n$, are staying in the **rows** of $\mathrm{mat}(f)$. This agreement is natural in view of the fact that the dual spaces are left ones.

1.9 The Euclidean spaces

In this Section we assume that the field $\mathbf{K}$ is provided by an **involution** $\alpha \mapsto \overline{\alpha}$ with the following properties:

i) $\overline{\overline{\alpha}} = \alpha$;

ii) $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$;

iii) $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$;

iv) $\alpha\overline{\alpha} = \overline{\alpha}\alpha$.

Hence, the involution is an anti-automorphism $\mathbf{K} \to \mathbf{K}$. As a consequence, $\overline{\alpha} \ne 0$ if $\alpha \ne 0$.

The subset of all fixed points of the involution,

$$\mathbf{k} = \{\lambda : \ \lambda \in \mathbf{K},\ \overline{\lambda} = \lambda\}, \tag{1.42}$$

is a subfield of $\mathbf{K}$. We suppose that the subfield $\mathbf{k}$ is **central**, i.e. $\lambda\alpha = \alpha\lambda$ for all $\lambda \in \mathbf{k}$, $\alpha \in \mathbf{K}$. Sometimes it is convenient to write down $x\lambda^{-1}$ with $\lambda \in \mathbf{k}$ as the quotient $\frac{x}{\lambda} \equiv x/\lambda$.

EXAMPLE 1.9.1. The subfield $\mathbf{k}$ is $\mathbf{R}$ for the classical fields $\mathbf{R},\ \mathbf{C},\mathbf{H}$.

If the characteristics of the field $\mathbf{K}$ is $\neq 2$ then any elements $\alpha\in\mathbf{K}$ is uniquely presented as

$$\alpha=\lambda+\mu,\quad \overline{\lambda}=\lambda,\quad \overline{\mu}=-\mu, \tag{1.43}$$

namely,

$$\lambda=\frac{\alpha+\overline{\alpha}}{2},\quad \mu=\frac{\alpha-\overline{\alpha}}{2}. \tag{1.44}$$

In the classical situation λ is the **real part** and μ is the **imagine part** of α,

$$\lambda=\Re\mathrm{e}(\alpha),\quad \mu=\Im\mathrm{m}(\alpha). \tag{1.45}$$

A subfield $\mathbf{L}\in\mathbf{K}$ is called **involutive** if it is invariant with respect to the involution, i.e.

$$\lambda\in\mathbf{L}\Rightarrow\overline{\lambda}\in\mathbf{L}. \tag{1.46}$$

The subfield $\mathbf{k}$ is trivially involutive, in particular, so is $\mathbf{R}$ in $\mathbf{C}$ or $\mathbf{H}$. The subfield $\mathbf{C}$ in $\mathbf{H}$ is also involutive.

Let E be a right linear space over $\mathbf{K}$, $\dim E=m$. The **inner product** $\langle x,y\rangle$ on E is a mapping $E\times E\to\mathbf{K}$ such that for all vectors $x,y\in E$ and all scalars $\alpha\in\mathbf{K}$ the following identities hold:

1) $\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle$

2) $\langle x\alpha,y\rangle=\overline{\alpha}\langle x,y\rangle$

3) $\overline{\langle x,y\rangle}=\langle y,x\rangle$

4) $\langle x,x\rangle\neq 0$ for $x\neq 0$.

Note that 3) implies

$$\overline{\langle x,x\rangle}=\langle x,x\rangle \tag{1.47}$$

for all $x\in E$, i.e. $\overline{\langle x,x\rangle}\in\mathbf{k}$.

It is trivially follows from 1-3 that

$$\langle x,y+z\rangle=\langle x,y\rangle+\langle x,z\rangle;\quad \langle x,y\alpha\rangle=\langle x,y\rangle\alpha \tag{1.48}$$

and

$$\langle x,0\rangle=0;\quad \langle 0,y\rangle=0 \tag{1.49}$$

for all $x,y\in E$.

Similarly, one can define an inner product in a left linear space. The only change in 2 is needed:

$$\langle\alpha x,y\rangle=\alpha\langle x,y\rangle. \tag{1.50}$$

A linear space provided with an inner product is called an **Euclidean space.** In this Section E means a right Euclidean space. The simplest example is $E=\mathbf{K}$ with $\langle\xi,\eta\rangle=\overline{\xi}\eta$.

The vectors $x,y\in E$ are called **orthogonal** if $\langle x,y\rangle=0$. We denote the orthogonality relation by $x\perp y$. In particular, (1.49) means that $x\perp 0,\ \ 0\perp y$. The main properties of the orthogonality relation are

a) $x \perp y \Rightarrow y \perp x$;

b) $x \perp x \Rightarrow x = 0$;

c) $x \perp y \Rightarrow y \perp x\alpha$;

d) $(x \perp y_1)\&(x \perp y_2) \Rightarrow x \perp (y_1 + y_2)$.

For any vector x and any nonempty set $Z \subset E$ we define the **orthogonality** $x \perp Z$ as $x \perp z$ for all $z \in Z$. It follows from c) and d) that

$$x \perp Z \Leftrightarrow x \perp \mathrm{Span}(Z). \tag{1.51}$$

In particular, for a subspace X we have that $x \perp X$ if and only if x is orthogonal to a basis of X.

PROPOSITION 1.9.2. *Let $X_1, \ldots, X_l$ be subspaces and let a vector x is orthogonal to each of them. Then x is orthogonal to the sum of these subspaces.*

Proof. This immediately follows from d).

The following statement is a relevant generalization of the **Pythagor Theorem**.

PROPOSITION 1.9.3. *If $x \perp y$ then*

$$\langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle. \tag{1.52}$$

Proof. This immediately follows from the basis identities for the inner product. □

A system $(v_k)_1^\nu$ is called **orthogonal** if $v_i \perp v_k, \quad i \neq k$. Therefore an orthogonal system is such that

$$\langle v_i, v_k \rangle = \delta_{ik} \langle v_i, v_i \rangle, \quad 1 \leq i, k \leq \nu. \tag{1.53}$$

For any system $(v_k)_1^\nu$ its **Gram matrix** is $[\langle v_i, v_k \rangle], \quad 1 \leq i, k \leq \nu$. *A system is orthogonal if and only if its Gram matrix is diagonal.* In particular, if $(v_k)_1^m$ is a basis and $[\gamma_{ik}]$ is its Gram matrix then the inner product can be expressed in the coordinate form

$$\langle x, y \rangle = \sum_{i,k=1}^{m} \overline{\xi_i} \gamma_{ik} \eta_k. \tag{1.54}$$

If the basis is orthogonal then

$$\langle x, y \rangle = \sum_{i=1}^{m} \overline{\xi_i} \gamma_{ii} \eta_i = \sum_{i=1}^{m} \gamma_{ii} \overline{\xi_i} \eta_i, \tag{1.55}$$

since $\gamma_{ii} = \langle v_i, v_i \rangle \in \mathbf{k}$.

An orthogonal system $(v_k)_1^\nu$ with $\langle v_k, v_k \rangle = 1, \ 1 \leq k \leq \nu$, is called **orthonormal**. In other words, this is a system such that

$$\langle v_i, v_k \rangle = \delta_{ik}, \quad 1 \leq i, k \leq \nu, \tag{1.56}$$

i.e. its Gram matrix is the unit matrix.

Conversely, given a $m \times m$-matrix $[\gamma_{ik}]$ such that

$$\gamma_{ki} = \overline{\gamma_{ik}} \tag{1.57}$$

and

$$\sum_{i,k=1}^{m} \overline{\xi}_i \gamma_{ik} \xi_k \neq 0 \tag{1.58}$$

for all columns $[\xi] \neq 0$, the formula (1.54) defines an inner product in the space $\mathbf{K}^m$ as well as in any m-dimensional right linear space where $[\xi]$, $[\eta]$ are the coordinate columns of vectors x, y for a basis. The corresponding Gram matrix is just $[\gamma_{ik}]$. In particular, for the **standard** inner product in $\mathbf{K}^m$,

$$\langle x, y\rangle = \sum_{k=1}^{m} \overline{\xi}_k \eta_k, \tag{1.59}$$

the canonical basis is orthonormal.

EXAMPLE 1.9.4.Consider the quaternionic space $\mathbf{H}^m$. Let $x, y \in \mathbf{H}^m$, i.e. $x = [\xi_i + \eta_i \mathbf{j}]_1^m$ and $y = [\zeta_i + \omega_i \mathbf{j}]_1^m$ with complex $\xi_i, \eta_i, \zeta_i, \omega_i$. According to (1.59), (1.2) and (1.3),

$$\langle x, y\rangle = \sum_{i=1}^{m} (\overline{\xi}_i - \eta_i \mathbf{j})(\zeta_i + \omega_i \mathbf{j}) = \sum_{i=1}^{m} (\alpha_i + \beta_i \mathbf{j}), \tag{1.60}$$

where

$$\alpha_i = \overline{\xi}_i \zeta_i + \eta_i \overline{\omega}_i, \quad \beta_i = \overline{\xi}_i \omega_i - \eta_i \overline{\zeta}_i. \tag{1.61}$$

As a consequence we obtain

$$|\langle x, y\rangle|^2 = \langle x, y\rangle \cdot \overline{\langle x, y\rangle} = \sum_{s,l=1}^{m} (\alpha_s + \beta_s \mathbf{j})(\overline{\alpha}_l - \beta_l \mathbf{j}) = \sum_{s,l=1}^{m} (\alpha_s \overline{\alpha}_l + \beta_s \overline{\beta}_l). \tag{1.62}$$

This formula will be effectively used in Section 3.3.

Note that in the left linear space $(\mathbf{K}^m)'$ the **standard** inner product is

$$\langle x, y\rangle = \sum_{k=1}^{m} \xi_k \overline{\eta}_k, \tag{1.63}$$

the canonical basis is orthonormal again.

PROPOSITION 1.9.5. *Any orthogonal system $(v_k)_1^\nu$ of nonzero vectors is linearly independent.*

Proof. Let

$$\sum_{k=1}^{\nu} v_k \alpha_k = 0$$

with some coefficients α_k. Then

$$0 = \sum_{k=1}^{\nu} \langle v_i, v_k \alpha_k\rangle = \sum_{k=1}^{\nu} \langle v_i, v_k\rangle \alpha_k = \sum_{k=1}^{\nu} \delta_{ik} \langle v_i, v_k\rangle \alpha_k = \langle v_i, v_i\rangle \alpha_i$$

for all $1 \le i \le \nu$. Since $\langle v_i, v_i\rangle \neq 0, \quad 1 \le i \le \nu$, all α_i are equal to zero. □

COROLLARY 1.9.6. *Any orthonormal system is linearly independent.*

The following statement is a key lemma to the further development of the Euclidean geometry.

LEMMA 1.9.7. *Let X be a subspace provided with an orthogonal basis $(e_k)_1^\nu$. Then for any vector v there exists a unique vector $u \in X$ such that*

$$v - u \perp X. \tag{1.64}$$

Such vector v is called an **orthogonal projection** of v on X and denoted by $u = \mathrm{pr}_X v$.

Proof. We look for a vector

$$u = \sum_{k=1}^{\nu} e_k \alpha_k \tag{1.65}$$

satisfying (1.64). The latter is equivalent to

$$(v - u) \perp e_i, \quad 1 \le i \le \nu,$$

or, equivalently,

$$\langle e_i, v - \sum_{k=1}^{\nu} e_k \alpha_k \rangle = 0, \quad 1 \le i \le \nu,$$

i.e.

$$\langle e_i, v \rangle = \sum_{k=1}^{\nu} \langle e_i, e_k \rangle \alpha_k = \sum_{k=1}^{\nu} \delta_{ik} \langle e_k, e_k \rangle \alpha_k = \alpha_i \langle e_i, e_i \rangle, \quad 1 \le i \le \nu.$$

Hence, u is an orthogonal projection of v on X if and only if in (1.65)

$$\alpha_k = \frac{\langle e_k, v \rangle}{\langle e_k, e_k \rangle}.$$

□

COROLLARY 1.9.8. *For any subspace $X \subset E$ and any vector v there exists a unique vector $\mathrm{pr}_X v$ such that*

$$\boxed{\mathrm{pr}_X v \in X, \quad (v - \mathrm{pr}_X v) \perp X} \quad . \tag{1.66}$$

Formula (1.69) yields $\mathrm{pr}_X v$ explicitly.

Now the following statement is obvious.

COROLLARY 1.9.9. $\mathrm{pr}_X v = v \Leftrightarrow v \in X$.

Proof. Since $v - v = 0 \perp X$, we obtain $\mathrm{pr}_X v = v$ in the case $v \in X$ by uniqueness in Corollary 1.9.8. Conversely, if $\mathrm{pr}_X v = v$ then $v \in X$ by (1.66). □

COROLLARY 1.9.10. *The equality*

$$\langle \mathrm{pr}_X v, \mathrm{pr}_X v\rangle = \langle v, v\rangle \tag{1.67}$$

holds if and only if $v \in X$.

Proof. By the Pythagor Theorem (1.66) implies

$$\langle v, v\rangle = \langle \mathrm{pr}_X v, \mathrm{pr}_X v\rangle + \langle v - \mathrm{pr}_X v, v - \mathrm{pr}_X v\rangle. \tag{1.68}$$

Hence, (1.67) is equivalent to the equality

$$\langle v - \mathrm{pr}_X v, v - \mathrm{pr}_X v\rangle$$

which, in turn, is equivalent to the $\mathrm{pr}_X v = v$ and the latter is equivalent to $v \in X$ by Corollary 1.9.9. □

A useful reformulation of Corollary 1.9.10 is

COROLLARY 1.9.11. *If some vectors* u *and* v *are such that*

$$\langle u, u\rangle = \langle v, v\rangle, \quad u \perp v - u$$

then $u = v$.

Proof. Taking $X = (v - u)^{\perp}$ we obtain $u = \mathrm{pr}_X v$. By Corollary 1.9.10 $v \in X$. Hence, $u = v$. □

As we have seen in the proof of Lemma 1.9.7,

$$\boxed{\mathrm{pr}_X v = \sum_{k=1}^{\nu} e_k \frac{\langle e_k, v\rangle}{\langle e_k, e_k\rangle}} \quad . \tag{1.69}$$

THEOREM 1.9.12. *There exists an orthogonal basis in the Euclidean space* E.

Proof. We use the induction on dimension. Note that for any one-dimensional space the statement is empty, so it is trivially true.

Consider a subspace X in E, $\dim X = m - 1$. By the induction assumption there exists an orthogonal basis $(e_k)_1^{m-1}$ in X. Taking a vector $v \notin X$ and setting

$$e_m = v - \mathrm{pr}_X v$$

we obtain the orthogonal basis $(e_k)_1^m$ in E. Indeed, $e_m \perp X$ and $e_m \neq 0$. □

REMARK 1.9.13. The above proof is actually a version of the **orthogonalization process**. In fact, Theorem 1.9.12 can be specified as follows: *there exists an orthogonal basis* $e_1, \ldots, e_m$ *which is compatible with a given filtration* $X_1 \subset X_2 \subset \ldots \subset X_m$. *Such a basis is unique up to proportionality*

$$e_k \mapsto e_k \lambda_k, \quad \lambda_k \neq 0, \quad 1 \le k \le m.$$

Indeed, if $(e'_k)_1^m$ is another such basis then e'_k belongs to X_k and it is orthogonal to X_{k-1}, $2 \le k \le m$. The decomposition of e'_k for the basis $(e_1, \ldots, e_k)$ in X_k is reduced to $e'_k = e_k \lambda_k$ with $\lambda_k = \langle e_k, e'_k\rangle / \langle e_k, e_k\rangle$. Obviously, $\lambda_k \neq 0$. □

Applying (1.69) to the case $X = E$ we obtain

COROLLARY 1.9.14. *For any $v \in E$ its decomposition for an orthogonal basis $(e_k)_1^m$ in E is*

$$v = \sum_{k=1}^{m} e_k \frac{\langle e_k, v\rangle}{\langle e_k, e_k\rangle}. \tag{1.70}$$

This is a finite-dimensional counterpart of the Fourier decomposition in Analysis. For this reason the values

$$\xi_k = \frac{\langle e_k, v\rangle}{\langle e_k, e_k\rangle}, \quad 1 \le k \le m. \tag{1.71}$$

are called the **Fourier coefficients.** The following example is classical.

EXAMPLE 1.9.15.Let Π be the complex (infinite-dimensional) linear space of all polynomials of one variable over $\mathbf{C}$. We can consider them as some complex-valued functions on a finite interval (a, b) of the real axis $\mathbf{R}$. Let $\omega(u)$ be a positive-valued continuous function on (a, b) such that

$$\int_a^b \omega(u)\, du < \infty. \tag{1.72}$$

The space Π provided with the inner product

$$\boxed{(p_1, p_2)_\omega = \int_a^b \overline{p_1(u)} p_2(u)\omega(u)\, du} \tag{1.73}$$

will be denoted by (Π, ω), the function ω in this context is called the **weight**. The weight ω is called **normalized** if

$$\int_a^b \omega(u)\, du = 1.$$

Obviously, for any weight ω the weight

$$\boxed{\Omega(u) = \frac{\omega(u)}{\int_a^b \omega(u)\, du}} \tag{1.74}$$

is normalized.

The finite-dimensional subspaces

$$\boxed{\Pi_d = \{p \in \Pi : \deg p \le d\}}\ , \quad \dim \Pi_d = d + 1, \quad d \in \mathbf{N}, \tag{1.75}$$

form a filtration of the whole space Π, i.e.

$$\Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \ldots, \quad \bigcup_{d=0}^{\infty} \Pi_d = \Pi. \tag{1.76}$$

Any sequence of polynomials $(p_k)_0^\infty$, $\deg p_k = k$, is a basis in Π which is compatible with filtration (1.76). By Remark 1.9.13 there exists such an orthogonal basis $(\widetilde{p}_k)_0^\infty$ in (Π, ω) and this is unique up to proportionality. One can say that we have the filtration (Π_d, ω), $d \in \mathbf{N}$, of the Euclidean space (Π, ω).

By the way, the most general construction of an inner product in functional space is

$$\boxed{(\phi_1, \phi_2) = \int_S \overline{\phi_1}\phi_2 \, d\mu} \tag{1.77}$$

where S is a nonempty set provided with a measure μ (so that (S, μ) is a measurable set) and the functions under consideration are measurable functions $\phi : S \to \mathbf{C}$ such that

$$\int_S |\phi|^2 \, d\mu < \infty.$$

The next corollaries of Theorem 1.9.12 are related to the orthogonal subspaces. The subspaces X and Y are called **orthogonal**, $X \perp Y$, if every vector in X is orthogonal to every vector in Y. In particular, so are X and $X^\perp$ where the latter is defined as the set of all vectors $y \perp X$. (It is clear that $X^\perp$ is a subspace.)

Obviously, $X \perp Y \Rightarrow X \cap Y = 0$. The direct sum of orthogonal subspaces X and Y is called the **orthogonal sum** and it is denoted by $X \oplus Y$.

COROLLARY 1.9.16. *There is the orthogonal decomposition*

$$\boxed{E = X \oplus X^\perp} \tag{1.78}$$

for any subspace $X \in E$.

Indeed, for any v we have $v = x + y$, where $x = \mathrm{pr}_X v \in X$ and $y = v - x \in X^\perp$.

Thus, $X^\perp$ is a direct complement of X. It is called the **orthogonal complement**. Its dimension is

$$\boxed{\dim X^\perp = \mathrm{codim} X} \; . \tag{1.79}$$

In particular,

$$0^\perp = E, \quad E^\perp = 0. \tag{1.80}$$

Moreover, formula (1.79) shows that

$$X \neq E \Leftrightarrow X^\perp \neq 0. \tag{1.81}$$

In other words, we have

COROLLARY 1.9.17. *$X \neq E$ if and only if the only vector $y \perp X$ is $y = 0$.*

(Cf. Proposition 1.6.4 and see Theorem 1.9.21 below for an additional comparison.)

An equivalent formulation of Corollary 1.9.17 is

COROLLARY 1.9.18. *A system of vectors $(v_k)_1^l$ is complete in E if and only if the only vector $y \perp v_k, \quad 1 \leq k \leq l$, is $y = 0$.*

One more consequence of Theorem 1.9.12 is

COROLLARY 1.9.19 $X^{\perp\perp} = X$.

Proof. First of all, $X \subset X^{\perp\perp}$ by definition of $X^{\perp\perp} = (X^{\perp})^{\perp}$. On the other hand,

$$\dim X^{\perp\perp} = \text{codim} X^{\perp} = \dim X.$$

□

Any system of subspaces $(X_k)_1^l$ is called **orthogonal** if $X_i \perp X_k, \quad 1 \leq i < k \leq l$.

PROPOSITION 1.9.20. *If the subspaces $X_1, \ldots, X_l$ are pairwise orthogonal then they are independent.*

Proof. Let

$$0 = \sum_{k=1}^{l} x_k, \quad x_k \in X_k.$$

Then for any k we get

$$0 = \langle x_i, 0 \rangle = \langle x_i, \sum_{k=1}^{l} x_k \rangle = \langle x_i, x_i \rangle,$$

whence $x_i = 0, \quad 1 \leq i \leq l$. □

Consider the dual space E' for the Euclidean space E. Obviously, for any $y \in E$ the scalar valued function $x \mapsto \langle y, x \rangle$ is a linear functional. An important statement is the following **Riesz Theorem.**

THEOREM 1.9.21. *For any linear functional $\varphi \in E'$ there exists a unique vector u such that*

$$\boxed{\varphi(x) = \langle u, x \rangle} \quad . \tag{1.82}$$

Thus, (1.82) is a general form of a linear functional on the Euclidean space E.

Proof. If $\varphi = 0$ then the only vector u satisfying (1.82) is $u = 0$. For $\varphi \neq 0$ we consider the hyperplane $X = \text{Ker}\varphi$. Then $\dim X^{\perp} = 1$. Consider a basis vector $w \in X^{\perp}$ and the corresponding linear functional

$$\psi(x) = \langle w, x \rangle.$$

Then

$$\text{Ker}\psi = (w)^{\perp} = (X^{\perp})^{\perp} = X^{\perp\perp} = X = \text{Ker}\varphi.$$

By Proposition 1.7.5 $\varphi = \lambda\psi$ with a scalar coefficient λ. Hence,

$$\varphi(x) = \lambda\psi(x) = \lambda\langle w, x\rangle.$$

We obtain (1.82) with $u = w\overline{\lambda}$.

To prove the uniqueness of u we observe that if $\varphi(x) = \langle u_1, x\rangle$ and $\varphi(x) = \langle u_2, x\rangle$ then $\langle u_1 - u_2, x\rangle = 0$ for all x. This means that $u_1 - u_2 \in E^{\perp} = 0$, i.e. $u_1 - u_2 = 0$, $u_1 = u_2$. □

A mapping $\vartheta : E \to \mathbf{K}$ is called a **semi-linear functional** on E if it is **additive**, i.e.

1) $\vartheta(x+y) = \vartheta\, x + \vartheta\, y \quad (x, y \in E)$

and **semi-homogeneous,**

2) $\vartheta(x\alpha) = \overline{\alpha}(\vartheta(x)) \quad (x \in E, \quad \alpha \in \mathbf{K})$.

The set of all semi-linear functionals on E is denoted by E^* and it is called the **conjugate space** to E. This is a right linear space over $\mathbf{K}$ like the dual space E'. (Recall that E' is a left linear space.) It is easy to see that the formula

$$\vartheta(x) = \overline{\varphi(x)}, \quad \varphi \in E', \tag{1.83}$$

defines a bijective mapping $E' \to E^*$, the inverse mapping is given by

$$\varphi(x) = \overline{\vartheta(x)}. \tag{1.84}$$

Obviously, if $\varphi \leftrightarrow \vartheta$ then $\alpha\varphi \leftrightarrow \vartheta\overline{\alpha}$. Taking into account (1.83) and (1.84) we obtain from Theorem 1.9.21 its $(*)$-version.

THEOREM 1.9.22. *A general form of a semi-linear functional on the Euclidean space E is*

$$\boxed{\vartheta(x) = \langle x, u\rangle} \;. \tag{1.85}$$

The correspondence between ϑ and u in (1.85) *is an isomorphism between E^* and E.*

Later on we identify E^* with E by this **canonical** isomorphism.

COROLLARY 1.9.23. $\dim E^* = \dim E$

The homomorphisms $f : E \to F$ (E is the Euclidean space under consideration, F is a right linear space) can be described in the following way (which is important for our purposes in the next Chapters).

Let $(v_k)_1^n$ be a fixed basis in F. Then the coordinates of fx are linear functionals of $x \in E$. The Riesz Theorem yields a system $(u_k)_1^n \subset E$ such that the decomposition

$$fx = \sum_{k=1}^{n} v_k\langle u_k, x\rangle \tag{1.86}$$

holds. The mapping $f \mapsto (u_k)_1^n$ from $\mathrm{Hom}(E, F)$ into E^n (the latter is the n-th Cartesian power) is bijective. For a homomorphism f we call the system $(u_k)_1^n$ in (1.86) the **frame** of f.

If F is also an Euclidean space and $f \in \mathrm{Hom}(E, F)$, the **conjugate** homomorphism $f^* : F \to E$ is defined by the identity

$$\langle fx, y\rangle_F = \langle x, f^*y\rangle_E. \tag{1.87}$$

In this context we have

PROPOSITION 1.9.24. *For any $f \in \mathrm{Hom}(E, F)$ there exists a unique conjugate $f^* \in \mathrm{Hom}(F, E)$.*

Proof. For any $y \in F$ we have the semi-linear functional $\vartheta_y(x) = \langle fx, y\rangle_F$ on E. By Theorem 1.9.22 there exists a unique vector $u \in E$ such that $\langle fx, y\rangle_F = \langle x, u\rangle_E$. Setting $u = f^*y$ we obtain (1.87), f^* is a mapping $F \to E$. As usual, one can check that f^* is a homomorphism. □

Here is a list of properties of the conjugate mapping $f \mapsto f^*$ which are similar to (1.36)-(1.39)

1) $f^{**} = f$;

2) $(f + g)^* = f^* + g^*$;

3) $(gf)^* = f^*g^*$;

4) $(\mathrm{id}_F)^* = \mathrm{id_F}$.

All the properties can be immediately established by (1.87). For example,

$$\langle x, f^{**}y\rangle_F = \langle f^*x, y\rangle_E = \langle x, fy\rangle_F$$

We also have

PROPOSITION 1.9.25. *Let f be a linear operator in E. Then the conjugate operator f^* in E is invertible if and only if f is invertible. In this situation*

$$\boxed{(f^*)^{-1} = (f^{-1})^*} \quad . \tag{1.88}$$

The proof is literally the same as for Proposition 1.8.3.

A linear operator f in E is called **unitary** if it is preserves the inner product, i.e.

$$\boxed{\langle fx, fy\rangle = \langle x, y\rangle} \tag{1.89}$$

for all $x, y \in E$. Further $U(E)$ stands for the set of unitary operators in E. The unitary operators in a real Euclidean space are called **orthogonal**. The corresponding notation for the group $U(E)$ is $O(E)$.

PROPOSITION 1.9.26. *The following conditions for $f \in L(E)$ are equivalent.*

(1) $f \in U(E)$;

(2) $f^*f = \mathrm{id}$;

(3) $ff^* = \mathrm{id}$;

(4) *f is invertible and $f^{-1} = f^*$.*

Proof. (1) $\Rightarrow$ (2). It follows from (1.89) that

$$\langle f^* f x, y\rangle = \langle x, y\rangle,$$

whence,

$$\langle (f^* f - \mathrm{id}) x, y\rangle = 0$$

for all x, y. Hence, $(f^* f - \mathrm{id})x = 0$ for all $x \in E$, so $f^* f = \mathrm{id}$.

(2) $\Rightarrow$ (3). If f is left-invertible and f^* is its left inverse then, by Proposition 1.7.12, f is right invertible with the right-inverse f^*.

(3) $\Rightarrow$ (4). This also immediately follows from Proposition 1.7.12.

(4) $\Rightarrow$ (1) We have

$$\langle fx, fy\rangle = \langle f^* f x, y\rangle = \langle f^{-1} f x, y\rangle = \langle \mathrm{id} x, y\rangle = \langle x, y\rangle.$$

□

PROPOSITION 1.9.27. *The set* $U(E)$ *is a subgroup of* $\mathrm{Aut}(E)$.

Proof. By Proposition 1.9.26 $U(E) \in \mathrm{Aut}(E)$. Obviously, if f and g preserve the inner product then so is gf. A similar argument works for f^{-1}. □

COROLLARY 1.9.28. *If* $f \in U(E)$ *then* $f^* \in U(E)$.

Let $(v_k)_1^\nu$ and $(u_k)_1^\nu$ be some systems. If there exists a unitary operator f such that

$$f v_k = u_k, \quad 1 \le k \le \nu \tag{1.90}$$

then this systems are called **unitary equivalent.** Obviously, (1.89) and (1.90) imply that the Gram matrices of unitary equivalent systems coincide, i.e. *the Gram matrix is unitary invariant.* A very important question is about completeness of this invariant. The answer is affirmative according to the classical Witt Theorem, see below.

LEMMA 1.9.29. *Suppose that the Gram matrices of systems* $(v_k)_1^\nu$ *and* $(u_k)_1^\nu$ *coincide. Let* $(v_k)_1^\rho$ *be a maximal linearly independent subsystem of* $(v_k)_1^\nu$. *If*

$$v_{\rho+j} = \sum_{k=1}^{\rho} v_k \alpha_{kj}, \quad 1 \le j \le \nu - \rho,$$

then

$$u_{\rho+j} = \sum_{k=1}^{\rho} u_k \alpha_{kj}, \quad 1 \le j \le \nu - \rho. \tag{1.91}$$

Proof. For $1 \le i \le \rho$ we have

$$\langle v_i, v_{\rho+j}\rangle = \langle v_i, \sum_{k=1}^{\rho} v_k \alpha_{kj}\rangle = \sum_{k=1}^{\rho} \langle v_i, v_k\rangle \alpha_{kj}.$$

Since the system $(u_k)_1^\nu$ has the same Gram matrix,

$$\langle u_i, u_{\rho+j}\rangle = \sum_{k=1}^{\rho}\langle u_i, u_k\rangle\alpha_{kj} = \langle u_i, \sum_{k=1}^{\rho} u_k\alpha_{kj}\rangle,$$

i.e,

$$\langle u_i, u_{\rho+j} - \sum_{k=1}^{\rho} u_k\alpha_{kj}\rangle = 0$$

for all i, $1 \le i \le \rho$. This means that

$$\sum_{k=1}^{\rho} u_k\alpha_{kj} = \mathrm{pr}_X u_{\rho+j} \tag{1.92}$$

where $X = \mathrm{Span}(u_1, \dots, u_\rho)$.

Now

$$\begin{aligned}\langle \mathrm{pr}_X u_{\rho+j}, \mathrm{pr}_X u_{\rho+j}\rangle &= \langle\sum_{k=1}^{\rho} u_k\alpha_{kj}, \sum_{k=1}^{\rho} u_k\alpha_{kj}\rangle = \sum_{i,k=1}^{\rho}\overline{\alpha}_{ij}\langle u_i, u_k\rangle\alpha_{kj} = \sum_{i,k=1}^{\rho}\overline{\alpha}_{ij}\langle v_i, v_k\rangle\alpha_{kj}\\ &= \langle\sum_{k=1}^{\rho} v_k\alpha_{kj}, \sum_{k=1}^{\rho} v_k\alpha_{kj}\rangle = \langle v_{\rho+j}, v_{\rho+j}\rangle = \langle u_{\rho+j}, u_{\rho+j}\rangle.\end{aligned}$$

By Corollary 1.9.10 $u_{\rho+j} \in X$ and (1.92) turns into (1.91). □

The key point of the whole proof is

LEMMA 1.9.30. *If $\nu < m$ and the linearly independent systems $(v_k)_1^\nu$ and $(u_k)_1^\nu$ have the same Gram matrices then there exists a vector w such that the extended systems $(v_k)_1^\nu \cup (w)$ and $(u_k)_1^\nu \cup (w)$ are both linearly independent and their Gram matrices coincide.*

Proof. Consider the subspace $X = \mathrm{Span}(v_1 - u_1, \dots, v_\nu - u_\nu)$ and take $w \in X^\perp$. If $r = \dim X$ then by (1.79)

$$\dim X^\perp = m - r. \tag{1.93}$$

It is already clear that the extended systems $(v_k)_1^\nu \cup (w)$ and $(u_k)_1^\nu \cup (w)$ have the same Gram matrices. Prove that w can be chosen in a way such that the systems $(v_k)_1^\nu \cup (w)$ and $(u_k)_1^\nu \cup (w)$ are both linearly independent. Otherwise, any $w \in X^\perp$ can be represented in form

$$w = \sum_{k=1}^{\nu} v_k\alpha_k = \sum_{k=1}^{\nu} u_k\alpha_k$$

according to Lemma 1.9.29. Thus,

$$X^\perp \subset Y \equiv \{w_{[\alpha]} = \sum_{k=1}^{\nu} v_k\alpha_k,\ \alpha_k \in A\}, \tag{1.94}$$

where

$$A = \{[\alpha] : \sum_{k=1}^{\nu}(v_k - u_k)\alpha_k = 0\}.$$

is a subspace of $\mathbf{K}^\nu$.

The set Y is the image of the homomorphism $\alpha \mapsto w_{[\alpha]}$ from A into $\mathrm{Span}(v_1, \dots, v_\nu)$. This is a monomorphism, since $v_1, \dots, v_\nu$ are linearly independent. As a result, $Y \approx A$, hence, $\dim Y = \dim A$. In order to calculate this dimension, we consider one more homomorphism, namely, $g : \mathbf{K}^\nu \to X$ defined as

$$g[\alpha] = \sum_{k=1}^{\nu} (v_k - u_k)\alpha_k.$$

Obviously, $A = \mathrm{Ker} g,\ X = \mathrm{Im} g$. By (1.24) $\dim A = \nu - r$, hence, also

$$\dim Y = \nu - r. \tag{1.95}$$

Comparing the dimensions (1.93) and (1.95) by (1.94) we obtain that $m - r \le \nu - r$ i.e. $\nu \ge m$, the contradiction. □

COROLLARY 1.9.31. *Two linearly independent systems with the same Gram matrices can be extended to some bases with the same Gram matrices.*

Proof. Let $(v_k)_1^\nu$ and $(u_k)_1^\nu$ be given systems and let $\mu = m - \nu$ so that $0 \le \mu \le m$. We use induction on μ. For $\mu = 0$ the statement is true obviously. Passing from μ to $\mu + 1$, i.e. from ν to $\nu - 1$ we can apply Lemma 1.9.30 and get a vector w such that the extended systems $(v_k)_1^{\nu-1} \cup (w)$ and $(u_k)_1^{\nu-1} \cup (w)$ are linearly independent and their Gram matrices coincide. By induction, the latter can be extended to some bases with the same Gram matrices. □

Now we are in position to prove the **Witt Theorem**.

THEOREM 1.9.32. *The systems $(v_k)_1^\nu$ and $(u_k)_1^\nu$ are unitary equivalent if and only if their Gram matrices coincide.*

Proof. Since the "if" part is trivial, we start with the equalities

$$\langle v_i, v_k \rangle = \langle u_i, u_k \rangle, \quad 1 \le i, k \le \nu.$$

If the system $(v_k)_1^\nu$ is a basis, $\nu = m$, we consider the linear operator f in E defined by conditions (1.90). Then for

$$x = \sum_{i=1}^{m} v_i \xi_i, \quad y = \sum_{k=1}^{m} v_k \eta_k$$

we get

$$\langle fx, fy \rangle = \sum_{i,k=1}^{m} \bar{\xi}_i \langle fv_i, fv_k \rangle \eta_k = \sum_{i,k=1}^{m} \bar{\xi}_i \langle u_i, u_k \rangle \eta_k = \sum_{i,k=1}^{m} \bar{\xi}_i \langle v_i, v_k \rangle \eta_k = \langle x, y \rangle,$$

i.e. f is an unitary operator.

If the system $(v_k)_1^\nu$ is not a basis then we consider its maximal linearly independent subsystem, say, $(v_k)_1^\rho$. The corresponding subsystem $(u_k)_1^\rho$ has the same Gram matrix and it is linearly independent by Lemma 1.9.29. By Corollary 1.9.31 both subsystems can be extended to some bases with the same Gram matrices. As we have already proved these bases are unitary equivalent. The corresponding unitary operator f satisfies

$$fv_{\rho+j} = u_{\rho+j}, \quad 1 \le j \le \nu - \rho$$

by Lemma 1.9.29 again. □

COROLLARY 1.9.33. *Any two orthonormal systems* $(v_k)_1^\nu$ *and* $(u_k)_1^\nu$ *are unitary equivalent.*

The existence of orthonormal systems depends on a property of the subfield $\mathbf{k} \subset \mathbf{K}$ (see 1.42).

THEOREM 1.9.34. *Suppose that all scalars* $\langle x, x\rangle$, $x \in E$, *are squares in* $\mathbf{k}$. *Then there exists an orthonormal basis in the Euclidean space* E.

Obviously, this theorem is applicable to all classical fields.

Proof. By Theorem 1.9.12 there exists an orthogonal basis $(e_k)_1^m$ in E. Let $v_k = \lambda_k^{-1} e_k$ where $\lambda_k^2 = \langle e_k, e_k\rangle$ and $\lambda_k \in \mathbf{k}$, so that $\overline{\lambda_k} = \lambda_k$. Then

$$\langle v_i, v_k\rangle = \lambda_i^{-1}\langle e_i, e_k\rangle\lambda_k^{-1} = \lambda_i^{-1}\delta_{ik}\lambda_k = \delta_{ik}.$$

□

Our assumption about $\mathbf{k}$ is substantial.

EXAMPLE 1.9.35.Take $\mathbf{K} = \mathbf{Q}$, the field of rational numbers; the involution is trivial, $\overline{\alpha} = \alpha$, so that $\mathbf{k} = \mathbf{K}$. Consider the 1-dimensional space $E = \mathbf{K}$ provided with the inner product $\langle\alpha, \beta\rangle = 2\alpha\beta$. Then $\langle\alpha, \alpha\rangle = 2\alpha^2$ for all $\alpha \in E$. □

As we know, the coordinate form of the inner product with respect to an orthonormal basis is standard,

$$\langle x, y\rangle = \sum_{k=1}^{m} \overline{\xi}_k \eta_k, \tag{1.96}$$

the Gram matrix is unit. In particular,

$$\langle x, x\rangle = \sum_{k=1}^{m} \overline{\xi}_k \xi_k. \tag{1.97}$$

Let (e_i) and (u_k) be some orthonormal bases in E and let $t = [\tau_{ik}]$ be the corresponding transition matrix,

$$u_k = \sum_{i=1}^{m} e_i \tau_{ik}.$$

By (1.96) the relation $\langle u_k, u_j\rangle = \delta_{kj}$ can be written down as

$$\sum_{i=1}^{m} \overline{\tau}_{ik}\tau_{ij} = \delta_{kj}. \tag{1.98}$$

This means that the system of columns of the matrix t is orthonormal with respect to the standard inner product in $\mathbf{K}^m$. The matrices with this property are called **unitary**. Thus, we have

PROPOSITION 1.9.36. *For any orthonormal basis* (e_i), *a basis* (u_k) *is orthonormal if and only if the matrix* t *of the transition* $(e_i) \to (u_k)$ *is unitary.*

COROLLARY 1.9.37. *The set* $U_m(\mathbf{K})$ *of unitary* $m \times m$ *matrices is a subgroup of the group* $GL_m(\mathbf{K})$ (see Corollary 1.5.5).

The standard notation for $U_m(\mathbf{R})$ and $U_m(\mathbf{C})$ is $U(m)$ and $O(m)$ respectively.

Given an orthonormal basis (e_i) in E, the mapping $f \mapsto \mathrm{mat}(f)$, $f \in L(E)$, preserves unitarity. Indeed, $f \in U(E)$ if and only if the system (fe_i) is orthonormal. (The "only if" part is trivial, the "if" part follows from linearity of f.) Thus, we have

PROPOSITION 1.9.38. *A linear operator f is unitary if and only if so is its matrix for an orthonormal basis.*

In such a way we obtain an isomorphism between the groups $U(E)$ and $U_m(\mathbf{K})$.

Now for any matrix $a = [\alpha_{ik}]$ let us introduce the conjugate matrix $a^* = [\alpha^*_{ik}]$,

$$\alpha^*_{ik} = \overline{\alpha_{ki}}, \quad 1 \leq i, k \leq m. \tag{1.99}$$

PROPOSITION 1.9.39. *For any orthonormal basis (e_i)*

$$\boxed{\mathrm{mat}(f^*) = (\mathrm{mat}(f))^*} \ . \tag{1.100}$$

Proof. By linearity the identity (1.87) (with $F = E$) is equivalent to

$$\langle fe_i, e_k\rangle = \langle e_i, f^* e_k\rangle, \quad 1 \leq i, k \leq m,$$

which is just (1.99). □

Combining Propositions 1.9.39 and 1.9.26 we obtain

PROPOSITION 1.9.40 *The following conditions for a matrix $a \in M_m(\mathbf{K})$ are equivalent*

(1) $a \in U_m(\mathbf{K})$;

(2) $a^* a = e$;

(3) $aa^* = e$;

(4) *a in invertible and $a^{-1} = a^*$.*

The equality (3) in entries says that

$$\sum_{k=1}^{m} \alpha_{ik}\overline{\alpha_{jk}} = \delta_{ij}, \quad 1 \leq i, j \leq m, \tag{1.101}$$

which means that the unitarity of a matrix can be defined as the orthonormality of the system of rows (cf.(1.63)).

1.10 Normed spaces

`In this Section the basis field` **K** `is supposed to be normed.` This means that **K** is provided with a **modulus** $|\cdot|$ which is a real-valued function on **K** with the properties

i) $|\alpha| > 0$ $(\alpha \neq 0)$, $|0| = 0$;

ii) $|\alpha\beta| = |\alpha| \cdot |\beta|$;

iii) $|\alpha + \beta| \leq |\alpha| + |\beta|$.

(As a consequence, $|1| = 1$, $|-\alpha| = |\alpha|$.)

Any normed field **K** is a metric (a fortiori, topological) space provided with the distance

$$\mathrm{dist}(\alpha, \beta) = |\alpha - \beta|. \tag{1.102}$$

The **triangle inequality** iii) implies the inequality

$$\Big||\alpha| - |\beta|\Big| \leq |\alpha - \beta|. \tag{1.103}$$

Hence, the modulus is a continuous function on **K**. Eventually, **K** *is a topological field.*

The classical fields **R**, **C**, **H** are normed with their standard modulae,

$$\boxed{|\alpha| = \sqrt{\overline{\alpha}\alpha}}\ , \tag{1.104}$$

where $\sqrt{\ }$ means the positive value of the square root. Note that

$$\boxed{|\alpha + \beta|^2 = |\alpha|^2 + 2\Re\mathrm{e}(\overline{\alpha}\beta) + |\beta|^2}\ . \tag{1.105}$$

Any subfield of a normed field is also normed by restriction of the modulus. In particular, **R** with its own norm is the normed subfield of the normed fields **C** and **H** and the same is true for $\mathbf{C} \subset \mathbf{H}$.

A right linear space E over **K** is called **normed** if it is provided with a **norm** $\|\cdot\|$ which is a real-valued function on E with the properties

a) $\|x\| > 0$, $(x \neq 0)$, $\|0\| = 0$;

b) $\|x\alpha\| = |\alpha|\|x\|$;

c) $\|x + y\| \leq \|x\| + \|y\|$,

which are natural analogues of i), ii), iii) respectively.

It is interesting to note that ii) follows from **b**) if $E \neq 0$. Indeed,

$$\|x\alpha\beta\| = \|x(\alpha\beta)\| = \|x\| \cdot |\alpha\beta|.$$

On the other hand,

$$\|x\alpha\beta\| = \|(x\alpha)\beta\| = \|x\alpha\| \cdot |\beta| = \|x\| \cdot |\alpha| \cdot |\beta|.$$

Thus, without the axiom ii) the theory of normed linear spaces is empty.

Any normed space E is a metric (a fortiori, topological) space provided with

$$\mathrm{dist}(x, y) = \|x - y\|. \tag{1.106}$$

This metric is **homogeneous**,

$$\mathrm{dist}(x\alpha, y\alpha) = |\alpha|\mathrm{dist}(x, y), \tag{1.107}$$

and **shift invariant**,

$$\mathrm{dist}(x + z, y + z) = \mathrm{dist}(x, y). \tag{1.108}$$

In particular, $\mathrm{dist}(x, 0) = \|x\|$. The function $\|x\|$ is continuous on E because of the inequality

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\|. \tag{1.109}$$

Formally a normed space is a pair $(E, \|\cdot\|)$ where E is a linear space and $\|\cdot\|$ is a norm on E. The subspaces of a normed space are automatically normed. All of them are closed since E is finite-dimensional (if $\mathbf{K}$ is locally compact).

The linear normed space $(E, \|\cdot\|)$ with the introduced metric is a linear topological space over the topological field $\mathbf{K}$, i.e. The basis operations (the multiplication by scalars and the addition of vectors) are continuous. This follows from the inequalities

$$\mathrm{dist}(x\alpha, y\beta) \leq |\alpha|\mathrm{dist}(x, y) + |\alpha - \beta|\mathrm{dist}(y, 0)$$

and

$$\mathrm{dist}(x' + y', x + y) \leq \mathrm{dist}(x' - x, 0) + \mathrm{dist}(y' - y, 0).$$

There are some important standard subsets in any linear normed space $(E, \|\cdot\|)$:

1. The **closed unit ball**

$$\mathbf{B(E)} \equiv B(E, \|\cdot\|) = \{x \in E, \|x\| \leq 1\}.$$

2. The **open unit ball**

$$\mathbf{B}_0(E) \equiv B_0(E, \|\cdot\|) = \{x \in E, \|x\| < 1\}.$$

3. The **unit sphere**

$$\mathbf{S}(E) \equiv \mathbf{S}(E, \|\cdot\|) = \{x \in E, \|x\| = 1\}.$$

Note that $\mathbf{B}(E)$ and $\mathbf{S}(E)$ are topologically closed while $\mathbf{B}_0(E)$ is topologically open.

The vectors $x \in \mathbf{S}(E)$ are called **normalized**. For any $z \in E$, $z \neq 0$, the vector $\widehat{z} = z/\|z\|$ is normalized. The mapping $z \mapsto \widehat{z}$ from $E/\{0\}$ onto $\mathbf{S}(E)$ is called the **normalization.**

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be some right linear normed spaces over the same basis field $\mathbf{K}$. A homomorphism $f : E \to F$ is called an **isometry** if

$$\|fx\|_F = \|x\|_E, \quad x \in E. \tag{1.110}$$

Obviously, $\mathrm{Ker} f = 0$ follows from (1.110), so that any isometry is a monomorphism. In other words, this is an **isometric embedding** of E into F (up to identification $E \equiv \mathrm{Im} f$).

Obviously, *for a chain of isometries*

$$E \xrightarrow{f} F \xrightarrow{g} G$$

the product gf is an isometry. If an isometry $f : E \to F$ is surjective (a fortiori, it is bijective) then f^{-1} is also an isometry.

For any normed space E the set $\mathrm{Iso}(E)$ *of all surjective isometries is a group.*

EXAMPLE 1.10.1. *The mapping $x \mapsto x\alpha$ with $|\alpha| = 1$ preserves the norm by virtue of* **b)**. *If the field* **K** *is commutative then this mapping is also a linear operator therefore this an isometry $E \to E$.* □

If a surjective isometry $E \to F$ does exist then F is called **isometric** to E. This relation is an equivalence, so that one can speak about a class of isometric spaces.

If $f : E \to F$ is an isometric embedding then $\mathrm{Im} f$ is isometric to E, so that E can be considered as a normed subspace of the normed space F.

The simplest example of a normed space is $E = \mathbf{K}$ with $\|\xi\| = |\xi|$, $\xi \in \mathbf{K}$. It follows from **b**) that all norms in 1-dimensional space are proportional. (They are proportional to $|\xi|$ if $E = \mathbf{K}$.)

Consider some standard norms on $\mathbf{K}^m$.

EXAMPLE 1.10.2. For

$$[\xi] = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}$$

the functional

$$\boxed{\|[\xi]\|_\infty = \max_{1 \le k \le m} |\xi_k|} \tag{1.111}$$

is a norm on $\mathbf{K}^m$. □

EXAMPLE 1.10.3. Let $p \ge 1$. The functional

$$\boxed{\|[\xi]\|_p = \left(\sum_{k=1}^{m} |\xi_k|^p \right)^{\frac{1}{p}}} \tag{1.112}$$

is called the ℓ_p**-norm** on $\mathbf{K}^m$. The triangle inequality in this case easily follows from iii) and classical Minkowski's inequality. □

Note that

$$\lim_{p \to \infty} \|[\xi]\|_p = \|\xi\|_\infty. \tag{1.113}$$

The normed space $(\mathbf{K}^m, \|\cdot\|_p)$, $1 \le p \le \infty$, is denoted by $\ell^m_{p;\mathbf{K}}$ (or briefly, ℓ^m_p in a context where $\mathbf{K}$ is fixed a priori).

If $n \leq N$ then ℓ_p^n can be considered as a subspace in ℓ_p^N in view of the **canonical isometric embedding** $\ell_p^n \to \ell_p^N$ given by

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \mapsto \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1.114}$$

THEOREM 1.10.4. *Let* $\mathbf{K} = \mathbf{R}, \mathbf{C}$ *or* $\mathbf{H}$. *Suppose that* q, p *are distinct and finite,* $m \geq 2$. *If there exists an isometric embedding* $\ell_q^m \to \ell_p^n$ *then* $q = 2$ *and* p *is an even integer.*

Proof. It sufficient to prove this for $m = 2$ because of canonical isometric embedding $\ell_q^2 \to \ell_q^m$. Let $f : \ell_q^2 \to \ell_p^n$ be an isometric embedding and let

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}. \tag{1.115}$$

Then we have the identity

$$\sum_{k=1}^{n} |\alpha\xi_k + \beta\eta_k|^p = (|\alpha|^q + |\beta|^q)^{\frac{p}{q}} \quad (\alpha, \beta \in \mathbf{K}). \tag{1.116}$$

Let ν be the number of non-zero entries ξ_k, $1 \leq \nu \leq n$. Without loss of generality one can assume that $\xi_k = 0$ for $\nu + 1 \leq k \leq n$. Then (1.116) with $\alpha = 1$ becomes

$$b_0|\beta|^p + \sum_{k=1}^{\nu} b_k|1 + \beta\gamma_k|^p = (1 + |\beta|^q)^{\frac{p}{q}} \tag{1.117}$$

where

$$b_k = |\xi_k|^p > 0, \quad \gamma_k = \eta_k\xi_k^{-1}, \quad b_0 = \sum_{k=\nu+1}^{n} |\eta_k|^p \geq 0. \tag{1.118}$$

For real β the identity (1.117) can be rewritten as

$$b_0|\beta|^p + \sum_{k=1}^{\nu} b_k(1 + 2\lambda_k\beta + \mu_k\beta^2)^{\frac{p}{2}} = (1 + |\beta|^q)^{\frac{p}{q}} \tag{1.119}$$

where

$$\lambda_k = \Re\mathrm{e}(\gamma_k), \quad \mu_k = |\gamma_k|^2, \tag{1.120}$$

see (1.105). For $q = 1$ the identity (1.119) is impossible since the right hand side of (1.119) turns out to be not differentiable at $\beta = 0$ in contrast to the left hand side because of $p \neq 1$ in this case. Therefore one can assume $q > 1$.

The identity (1.119) yields

$$b_0\beta^p + \sum_{k=1}^{\nu} b_k(1+2\lambda_k\beta+\mu_k\beta^2)^{\frac{p}{2}} = (1+\beta^q)^{\frac{p}{q}}, \quad \beta > 0. \tag{1.121}$$

Further the only identity (1.121) will be used so, one can assume that the functions $1+2\lambda_k\beta+\mu_k\beta^2$ are distinct, otherwise ν could be reduced. Now we extend the function β^q into the domain $\mathbf{C}\setminus\mathbf{R}_-$, where $\mathbf{R}_-$ is the negative real semiaxis,

$$\beta^q = \exp\left(q(\ln|\beta| + \mathbf{i}\mathrm{Arg}\beta)\right), \quad |\mathrm{Arg}\beta| < \pi.$$

This is a single-valued analytical extension. On this base we define a single-valued analytical branch

$$(1+\beta^q)^{\frac{p}{q}} = 1 + \frac{p}{q}\beta^q + \frac{1}{2!}\frac{p}{q}\left(\frac{p}{q}-1\right)\beta^{2q} + \dots, \quad |\beta| < 1, \quad \beta \notin [-1, 0]. \tag{1.122}$$

This is also a single-valued analytic branch for the left hand side of (1.121) in the same domain.

Suppose that $\frac{p}{q}$ is not integer. Then the singularities on the unit circle of the function (1.122) are roots of equation $1+\beta^q = 0$, i.e. they are

$$\beta_l = \exp\left(\frac{(2l+1)\pi\mathbf{i}}{q}\right), \ l \in \mathbf{Z}, \quad |2l+1| < q.$$

In particular, there is the singularity β_0. By identity (1.121) β_0 must be a root of one of polynomials $1+2\lambda_k\beta+\mu_k\beta^2$, $1 \le k \le \nu$. Let for definiteness

$$1 + 2\lambda_1\beta_0 + \mu_1\beta_0^2 = 0.$$

The second root is $\overline{\beta}_0 = \beta_{-1} \neq \beta_0$. Hence,

$$\mu_1 = \frac{1}{\overline{\beta}_0\beta_0} = 1, \quad \lambda_1 = -\frac{\beta_0+\overline{\beta}_0}{2} = -\cos\frac{\pi}{q}.$$

Hence, β_0 can not be a root of a polynomial $1+2\lambda_k\beta+\mu_k\beta^2$ with $k \ge 2$, otherwise this polynomial would be equal to $1+2\lambda_1\beta+\mu_1\beta^2$.

Now the identity (1.121) yields

$$(\beta-\beta_0)^{\frac{p}{2}}\phi(\beta) + \psi(\beta) = (\beta-\beta_0)^{\frac{p}{q}}\theta(\beta) \tag{1.123}$$

where ϕ, ψ, θ are single-valued analytic functions in a neighborhood of β_0 and, moreover, $\phi(\beta_0) \neq 0$, $\theta(\beta_0) \neq 0$. This implies that $q = 2$. Indeed, let $q > 2$. Then we rewrite (1.123) as

$$\theta(\beta) = (\beta-\beta_0)^{\frac{p}{2}-\frac{p}{q}}\phi(\beta) + (\beta-\beta_0)^{-\frac{p}{q}}\psi(\beta). \tag{1.124}$$

This is possible for only $\psi(\beta) \equiv 0$, otherwise $\theta(\beta_0) = 0$ or $\theta(\beta) \to \infty$ as $\beta \to \beta_0$ since $\frac{p}{q} \notin \mathbf{N}$ by assumption. But then we get $\theta(\beta_0) = 0$ from (1.124) since $\frac{p}{2} > \frac{p}{q}$. By contradiction we conclude that $q \le 2$. If $q < 2$ then $\frac{p}{2}$ turns out to be integer like $\frac{p}{q}$ before. As a result, we have a single-valued function on the left hand side of (1.121) but multi-valued one on the right hand side. Thus, $q = 2$.

Now the identity (1.121) becomes

$$b_0\beta^p + \sum_{k=1}^{\nu} b_k(1 + 2\lambda_k\beta + \mu_k\beta^2)^{\frac{p}{2}} = (1 + \beta^2)^{\frac{p}{2}} \tag{1.125}$$

with $\frac{p}{2} \notin \mathbf{N}$. In this case $\beta_0 = \mathbf{i}$, $\lambda_1 = 0$, $\mu_1 = 1$. Therefore $\nu = 1$, otherwise there would be some singularities different from $\pm\mathbf{i}$ on the left hand side of (1.125). The latter is reduced to

$$b_0\beta^p + b_1(1 + \beta^2)^{\frac{p}{2}} = (1 + \beta^2)^{\frac{p}{2}},$$

whence, obviously, $b_0 = 0$, $b_1 = 1$. For $k \geq 2$ we get $\xi_k = 0$ (since $\nu = 1$) and $\eta_k = 0$ (because of $b_0 = 0$ and (1.118)). This means that $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ are linearly dependent which is a contradiction since f is an isometry. We have proved that $\frac{p}{q}$ is an integer.

Now we compare the expansion (1.122) to the power expansion for the left hand side of (1.121). Note that $p \geq 2q$ (since $p \neq q$, $\frac{p}{q} \in \mathbf{N}$). Moreover $q > 1$ so, $p > 2$. Hence,

$$\sum_{k=1}^{\nu} b_k + p\left(\sum_{k=1}^{\nu} b_k\lambda_k\right)\beta + \frac{p}{2}\sum_{k=1}^{\nu} b_k\left(\mu_k + (p-2)\lambda_k^2\right)\beta^2 + o(\beta^2) = 1 + \frac{p}{q}\beta^q + o(\beta^2) \tag{1.126}$$

as β tends to zero.

Obviously,

$$\sum_{k=1}^{\nu} b_k\left(\mu_k + (p-2)\lambda_k^2\right) \geq 0 \tag{1.127}$$

but, in fact, the equality in (1.127) is impossible. Indeed, otherwise $\mu_1 = \lambda_1 = 0$ and $\nu = 1$, so that

$$b_0\beta^p + b_1 = (1 + \beta^q)^{\frac{p}{q}} = 1 + \frac{p}{q}\beta^q + \ldots$$

by (1.121) and (1.122). This contradicts the condition $p \neq q$.

With strong inequality in (1.127) we conclude that $q = 2$ in (1.126) and then p is even since $\frac{p}{q}$ is integer. $\square$

1.11 Euclidean norms

In this Section $\mathbf{K} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$, so that $|\alpha|$ is given by (1.104). This equality is a one-dimensional model for the definition of the norm

$$\boxed{\|x\| = \sqrt{\langle x, x\rangle}} \tag{1.128}$$

in an Euclidean space E over $\mathbf{K}$. This definition requires that

$$\langle x, x\rangle > 0 \quad (x \neq 0) \tag{1.129}$$

instead of the weaker inequality $\langle x, x\rangle \neq 0 \quad (x \neq 0)$ introduced in Section 1.9. However, in general the inner product $\langle x, y\rangle$ can be changed for $-\langle x, y\rangle$ so, the condition (1.129) can't be valid. But this change of sign will be prohibited if $\langle e, e\rangle > 0$ for some $e \neq 0$. The latter can be supposed without loss of generality, since $(\langle e, e\rangle)^{-1}\langle x, y\rangle$ is also an inner product.

LEMMA 1.11.1. *If $\langle e, e\rangle > 0$ for some $e \neq 0$ then (1.129) is valid.*

Proof. Let $\langle x, x\rangle < 0$ for some $x \in E$. Consider the function (a quadratic polynomial)

$$f(t) = \langle te + (1-t)x, te + (1-t)x\rangle, \quad t \in \mathbf{R}.$$

Then $f(0) = \langle x, x\rangle < 0$ and $f(1) = \langle e, e\rangle > 0$. By continuinity of f there exists $t_0 \in (0, 1)$ such that $f(t_0) = 0$. Hence, $t_0 e + (1 - t_0)x = 0$, i.e.

$$x = \frac{t_0}{t_0 - 1} e.$$

Then

$$\langle x, x\rangle = \frac{t_0^2}{(t_0 - 1)^2}\langle e, e\rangle > 0,$$

the contradiction. □

`We accept (1.129) and (1.128) in sequel.` Then we have

THEOREM 1.11.2. *Any Euclidean space is normed according to (1.128).*

In order to proof this we need the **Schwartz's inequality**

$$\boxed{|\langle x, y\rangle| \leq \|x\| \cdot \|y\|} \quad . \tag{1.130}$$

Let us prove the latter.

Without loss of generality one can assume that $\langle x, y\rangle \neq 0$ and $\|x\| = 1$. Indeed, (1.130) is homogeneous in x and the normalization $x \mapsto \widehat{x}$ preserves it; conversely, one can lift (1.130) from $\widehat{x}$ to x. Similarly, one can assume that $\langle x, y\rangle > 0$ passing from y to $y\lambda$ where $\lambda = |\langle x, y\rangle|\langle x, y\rangle^{-1}$.

Now let w be the orthogonal projection of y on Span(x), i.e. $w = x\langle x, y\rangle$ according to (1.69). Let $z = y - w$ so, $z \perp w$. By the Pythagor Theorem

$$\|y\|^2 = \|z\|^2 + \|w\|^2,$$

whence

$$\langle x, y\rangle = \|w\| \leq \|y\|.$$

It is easy to see that (1.130) turns into an equality if and only if the vectors x, y are linearly dependent.

Proof of Theorem 1.11.2. The norm properties **a)** and **b)** immediately follow from the properties of the inner product. The triangle inequality follows from (1.130):

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re\mathrm{e}\langle x, y\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2.$$

□

Thus, one can say that (1.128) is the **Euclidean norm** in the Euclidean space E. For this norm

$$\operatorname{dist}(x,y)=\sqrt{\|x\|^2+\|y\|^2-2\Re\mathrm{e}\langle x,y\rangle} \tag{1.131}$$

according to the general definition (1.102). In particular,

$$\operatorname{dist}(x,y)=\sqrt{2(1-\Re\mathrm{e}\langle x,y\rangle)}\quad (\|x\|=\|y\|=1). \tag{1.132}$$

In this situation, $|\Re\mathrm{e}\langle x,y\rangle|\leq 1$ by Schwartz's inequality, so that the angle $\vartheta=\arccos\Re\mathrm{e}\langle x,y\rangle$ is well-defined. In general, one can introduce

$$\vartheta=\arccos\frac{\Re\mathrm{e}\langle x,y\rangle}{\|x\|\cdot\|y\|}\quad (x\neq 0,\quad y\neq 0). \tag{1.133}$$

This $\vartheta=\vartheta(x,y)$ is called the **angle between vectors** x and y. Formula (1.132) can be rewritten as

$$\operatorname{dist}(x,y)=2\sin\frac{\vartheta(x,y)}{2}. \tag{1.134}$$

By the way, the angle $\vartheta(x,y)$ can be also considered as a distance, the **spherical distance**.

The standard coordinate realization of the Euclidean space is $\ell^m_{2;\mathbf{K}}$ with the inner product

$$\langle[\xi],[\eta]\rangle=\sum_{k=1}^m\overline{\xi_k}\eta_k. \tag{1.135}$$

The unit sphere $\mathbf{S}(\ell^m_{2;\mathbf{R}})$ is the standard m-dimensional sphere $\mathbf{S}^{m-1}$.

By (1.129) Theorem 1.9.34 provides an orthonormal basis in the Euclidean space E. Combining (1.128) and (1.97) we obtain the standard formula for the Euclidean norm

$$\boxed{\|x\|_2=\sqrt{\sum_{k=1}^m\overline{\xi_k}\xi_k}=\sqrt{\sum_{k=1}^m|\xi_k|^2}} \tag{1.136}$$

in coordinates with respect to an orthonormal basis.

Formula (1.136) shows that *all m-dimensional Euclidean spaces are isometric to ℓ^m_2*, the isometry $E\to\ell^m_2$ is $x\to[\xi]$ (cf. Theorem 1.7.13). Thus, the space ℓ^m_2 can be considered as the canonical representative of all m-dimensional Euclidean spaces, the **arithmetic m-dimensional Euclidean space.**

Note that the fields $\mathbf{C}$ and $\mathbf{H}$ are real Euclidean spaces and also $\mathbf{H}$ is a complex Euclidean space, the inner product is $\langle\alpha,\beta\rangle=\overline{\alpha}\beta$ in all cases. The corresponding Euclidean norm is the standard modulus $|\alpha|$ (cf. (1.104) and (1.128)). One can identify $\mathbf{C}$ and $\mathbf{H}$ with $\ell^2_{2;\mathbf{R}}$ and $\ell^4_{2;\mathbf{R}}$ respectively and $\mathbf{H}$ with $\ell^2_{2;\mathbf{C}}$ as well.

The realification $E_{\mathbf{R}}$ of an Euclidean space E can be considered as a real Euclidean space with respect to the inner product

$$\boxed{\langle x,y\rangle_{\mathbf{R}}=\Re\mathrm{e}\langle x,y\rangle}\ . \tag{1.137}$$

In particular, $\langle x,x\rangle_{\mathbf{R}} = \langle x,x\rangle$, i.e. $\|x\|_{\mathbf{R}} = \|x\|$. For this reason *the unit spheres* $\mathbf{S}(E)$ *and* $\mathbf{S}(E_{\mathbf{R}})$ *coincide*, so that $\mathbf{S}(E) = \mathbf{S}^{\delta m-1}$ where $\delta = [\mathbf{K} : \mathbf{R}]$, i.e. $\delta = 1, 2, 4$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ respectively. Also note that

$$U(E) \subset O(E_{\mathbf{R}}) \tag{1.138}$$

because of (1.137).

The unitary operators in E can be characterized as those orthogonal operators in $E_{\mathbf{R}}$ which are linear operators in E, i.e. they commute with the mapping $x \mapsto x\lambda,\ \ \lambda \in \mathbf{K}$ (being additive a priori). The latter commutation condition is valid as soon as is valid for $\lambda = \mathbf{i}$ over $\mathbf{C}$ or $\lambda = \mathbf{i}, \mathbf{j}, \mathbf{k}$ over $\mathbf{H}$.

PROPOSITION 1.11.3. *If E and F are Euclidean spaces and $f : E \to F$ is an isometry then*

$$\langle fx, fy\rangle = \langle x, y\rangle \tag{1.139}$$

for all $x, y \in E$.

Proof. The identity

$$\|y + x\lambda\|^2 = \|y\|^2 + |\lambda|^2\|x\|^2 + \overline{\lambda}\langle x, y\rangle + \langle y, x\rangle\lambda. \tag{1.140}$$

Therefore the expression $\overline{\lambda}\langle x, y\rangle + \langle y, x\rangle\lambda$ is invariant for $x \mapsto fx$, $y \mapsto fy$ where f is an isometry. Thus,

$$(\langle fy, fx\rangle - \langle y, x\rangle)\lambda = \overline{\lambda}(\langle x, y\rangle - \langle fx, fy\rangle)$$

or

$$\mu\lambda = -\overline{\lambda}\overline{\mu}, \quad \mu = \langle x, y\rangle - \langle fx, fy\rangle.$$

Putting $\lambda = \overline{\mu}$ we get $|\mu|^2 = -|\mu|^2$, i.e. $\mu = 0$ which yields (1.139). $\square$

COROLLARY 1.11.4. *If E is an Euclidean space then*

$$\boxed{\mathrm{Iso}(E) = U(E)}\ . \tag{1.141}$$

A *normed* space F is called **Euclidean** if there exists an inner product in F so that the corresponding Euclidean norm coincides with the initial norm on F.

EXAMPLE 1.11.5 Let $f : \ell_2^m \to \ell_p^n$ be an isometric embedding. Then $F = \mathrm{Im} f$ is an Euclidean subspace of the normed space ℓ_p^n. Conversely, if a subspace $F \subset \ell_p^n$ is Euclidean then there exists an isometric embedding $f : \ell_2^m \to \ell_p^n$, $\mathrm{Im} f = F$. $\square$

The problem of existence of isometric embeddings $\ell_2^m \to \ell_p^n$ is one of principal topic in the present work. We will come back to it in Section 4.7 after a development of a general cubature formulas theory.

Chapter 2

Some analytical preliminaries

2.1 Jacobi polynomials

Let us start with a preliminary information which partially can be found in [54, Chapter IV]. First of all, we recall that the classical **Jacobi polynomial** is the k-th member of the sequence $\left(P_k^{(\alpha,\beta)}(u)\right)_{k=0}^{\infty}$ of polynomials which are orthogonal on $[-1,1]$ with respect to the **Jacobi weight**

$$\boxed{\omega_{\alpha,\beta}(u) = (1-u)^{\alpha}(1+u)^{\beta} \quad (\alpha,\beta > -1)} \tag{2.1}$$

or,equivalently, to the **normalized Jacobi weight**

$$\boxed{\Omega_{\alpha,\beta}(u) = \frac{\omega_{\alpha,\beta}(u)}{\tau_{\alpha,\beta}}; \quad \tau_{\alpha,\beta} = \int_{-1}^{1} \omega_{\alpha,\beta}(u)\,du} \quad . \tag{2.2}$$

An explicit expression for Jacobi polynomials is

$$P_k^{(\alpha,\beta)}(u) = \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{\alpha+k}{\nu}\binom{\beta+k}{k-\nu}(u-1)^{k-\nu}(u+1)^{\nu}, \tag{2.3}$$

see [8]. Obviously,

$$\deg P_k^{(\alpha,\beta)} = k, \quad P_k^{(\alpha,\beta)}(1) = \binom{\alpha+k}{k} \tag{2.4}$$

and

$$P_k^{(\beta,\alpha)}(-u) = (-1)^k P_k^{(\alpha,\beta)}(u). \tag{2.5}$$

In particular, *the polynomials* $P_k^{(\alpha,\alpha)}(u)$ *are even for even* k *and odd for odd* k. The latter polynomials are in essence the Gegenbauer polynomials. More precisely, the **Gegenbauer (ultraspherical) polynomial** is defined as

$$C_k^\nu(u) = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(2\nu+k)}{\Gamma(2\nu)\Gamma(\nu+k+\frac{1}{2})} P_k^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(u), \quad \nu > -\frac{1}{2}, \tag{2.6}$$

so that

$$\deg C_k^\nu = k, \quad C_k^\nu(1) = \binom{2\nu+k-1}{k}. \tag{2.7}$$

In addition, by (2.5) and (2.7)

$$C_k^\nu(-1) = (-1)^k \binom{2\nu+k-1}{k}. \tag{2.8}$$

With a fixed ν the Gegenbauer polynomials $(C_k^\nu(u))_{k=0}^\infty$ are orthogonal on $[-1,1]$ with respect to the weight

$$\omega_{\nu-\frac{1}{2},\nu-\frac{1}{2}}(u) = (1-u^2)^{\nu-\frac{1}{2}}.$$

We especially need in the Gegenbauer polynomials with $\nu = \frac{q-2}{2}$, $q \in \mathbf{N}$, $q \geq 1$. They are orthogonal with respect to the weight

$$\boxed{\omega_q(u) = \omega_{\frac{q-3}{2},\frac{q-3}{2}}(u) = (1-u^2)^{\frac{q-3}{2}}} \tag{2.9}$$

or, equivalently, to

$$\boxed{\Omega_q(u) = \frac{\omega_q(u)}{\tau_q}} \tag{2.10}$$

where

$$\boxed{\tau_q = \int_{-1}^1 \omega_q(u)\,du = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}}\;. \tag{2.11}$$

The **Cristoffel-Darboux kernel** which relates to the Jacobi polynomials is

$$K_t^{(\alpha,\beta)}(u,v) = \sum_{k=0}^t \frac{P_k^{(\alpha,\beta)}(u)P_k^{(\alpha,\beta)}(v)}{\|P_k^{(\alpha,\beta)}\|^2_{\omega_{\alpha,\beta}}}. \tag{2.12}$$

According to the **Cristoffel-Darboux Formula**

$$\begin{aligned} K_t^{(\alpha,\beta)}(u,v) &= \frac{1}{2^{\alpha+\beta}(2t+\alpha+\beta+2)} \cdot \frac{\Gamma(t+2)\Gamma(t+\alpha+\beta+2)}{\Gamma(t+\alpha+1)\Gamma(t+\beta+1)} \\ &\times \frac{P_{t+1}^{(\alpha,\beta)}(u)P_t^{(\alpha,\beta)}(v) - P_t^{(\alpha,\beta)}(u)P_{t+1}^{(\alpha,\beta)}(v)}{u-v}. \end{aligned} \tag{2.13}$$

(Note that (2.13) can be extended to $u = v$ by passing to limit.) An important particular case is

$$K_t^{(\alpha,\beta)}(u) \equiv K_t^{(\alpha,\beta)}(u,1) = \frac{1}{2^{\alpha+\beta+1}} \cdot \frac{\Gamma(t+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(t+\beta+1)} P_t^{\alpha+1,\beta}(u), \tag{2.14}$$

whence

$$K_t^{(\alpha,\beta)}(1) = \frac{1}{2^{\alpha+\beta+1}} \cdot \frac{\Gamma(t+\alpha+\beta+2)\Gamma(t+\alpha+2)}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(t+1)\Gamma(t+\beta+1)}. \tag{2.15}$$

In fact, we need to calculate the quantity

$$\boxed{\Lambda_t^{(\alpha,\beta)} = 2^{\varepsilon} \tau_{\alpha,\beta} K_{[\frac{t}{2}]}^{(\alpha,\beta+\varepsilon)}(1)} \tag{2.16}$$

where $\varepsilon = \varepsilon_t = \mathrm{res}(t) (\mathrm{mod}\ 2)$ and

$$\boxed{\tau_{\alpha,\beta} = \int_{-1}^{1} \omega_{\alpha,\beta}(u)\,du = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}} \quad . \tag{2.17}$$

By substitution from (2.15) and (2.17) into (2.16) we obtain

$$\boxed{\Lambda_t^{(\alpha,\beta)} = \frac{\Gamma(\beta+1)\Gamma(\left[\frac{t}{2}\right]+\alpha+\beta+\varepsilon+2)\Gamma(\left[\frac{t}{2}\right]+\alpha+2)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha+2)\Gamma(\left[\frac{t}{2}\right]+1)\Gamma(\left[\frac{t}{2}\right]+\beta+\varepsilon+1)}} \quad . \tag{2.18}$$

The following specialization is the most important in the sequel.

THEOREM 2.1.1. *Let* $m \in \mathbf{N},\ m \geq 1$, *and let*

$$\Lambda_{\mathbf{K}}(m,t) = \Lambda_t^{(\alpha,\beta)}, \quad \alpha = \frac{\delta m - \delta - 2}{2}, \beta = \frac{\delta - 2}{2}, \tag{2.19}$$

where $\delta = [\mathbf{K} : \mathbf{R}]$. *Then*

$$\boxed{\Lambda_{\mathbf{R}}(m,t) = \binom{m+t-1}{m-1}} \tag{2.20}$$

and

$$\boxed{\Lambda_{\mathbf{C}}(m,t) = \binom{m+[\frac{t}{2}]-1}{m-1} \cdot \binom{m+[\frac{t+1}{2}]-1}{m-1}} \quad , \tag{2.21}$$

and

$$\boxed{\Lambda_{\mathbf{H}}(m,t) = \frac{1}{2m-1}\binom{2m+[\frac{t}{2}]-2}{2m-2}\cdot\binom{2m+[\frac{t+1}{2}]-1}{2m-2}}\quad . \tag{2.22}$$

Proof. By (2.19) and (2.18) we get

$$\Lambda_{\mathbf{K}}(m,t) = \frac{\Gamma(\frac{\delta}{2})\Gamma(\left[\frac{t}{2}\right]+\frac{\delta}{2}m+\varepsilon)\Gamma(\left[\frac{t}{2}\right]+\frac{\delta}{2}m-\frac{\delta}{2}+1)}{\Gamma(\frac{\delta}{2}m)\Gamma(\frac{\delta}{2}m-\frac{\delta}{2}+1)\Gamma(\left[\frac{t}{2}\right]+1)\Gamma(\left[\frac{t}{2}\right]+\frac{\delta}{2}+\varepsilon)}. \tag{2.23}$$

If $\mathbf{K}=\mathbf{R}$, i.e. $\delta=1$, then

$$\begin{aligned}\Lambda_{\mathbf{R}}(m,t) &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{t+\varepsilon}{2}+\frac{1}{2}m)\Gamma(\frac{t-\varepsilon}{2}+\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{t-\varepsilon}{2}+1)\Gamma(\frac{t+\varepsilon}{2}+\frac{1}{2})} \\ &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{t+\varepsilon}{2}+\frac{1}{2})}\cdot\frac{\Gamma(\frac{t+\varepsilon}{2}+\frac{1}{2}m)}{\Gamma(\frac{1}{2}m)}\cdot\frac{\Gamma(\frac{t-\varepsilon}{2}+\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}m+\frac{1}{2})}\cdot\frac{1}{(\frac{t-\varepsilon}{2})!}\end{aligned}$$

Because of the classical formula

$$\Gamma(u+k) = \Gamma(u)\prod_{i=0}^{k-1}(u+i), \quad k\in\mathbf{N}, \tag{2.24}$$

we get

$$\begin{aligned}\Lambda_{\mathbf{R}}(m,t) &= \frac{\prod_{i=0}^{\frac{t+\varepsilon}{2}-1}(\frac{m}{2}+i)\prod_{i=0}^{\frac{t-\varepsilon}{2}-1}(\frac{m+1}{2}+i)}{\prod_{i=0}^{\frac{t+\varepsilon}{2}-1}(\frac{1}{2}+i)\left(\frac{t-\varepsilon}{2}\right)!} = \frac{\prod_{i=0}^{t+\varepsilon-2}(m+i)\prod_{i=0}^{t-\varepsilon-2}(m+1+i)}{(t+\varepsilon-1)!!(t-\varepsilon)!!} \\ &= \frac{m(m+1)\cdot\ldots\cdot(m+t-1)}{t!} = \binom{m+t-1}{m-1}.\end{aligned}$$

For $\mathbf{K}=\mathbf{C}$ or $\mathbf{H}$, i.e. $\delta=2$ or 4, $\chi=\frac{\delta}{2}$, formula (2.23) becomes

$$\begin{aligned}\Lambda_{\mathbf{K}}(m,t) &= \frac{(\frac{t+\varepsilon}{2}+\chi m-1)!(\frac{t-\varepsilon}{2}+\chi m-\chi)!}{(\chi m-1)!(\chi m-\chi)!(\frac{t-\varepsilon}{2})!(\frac{t+\varepsilon}{2}+\chi-1)!} \\ &= \frac{(\chi m-\chi)!}{(\chi m-1)!}\cdot\frac{(\frac{t-\varepsilon}{2}+\chi m-\chi)!}{(\frac{t-\varepsilon}{2})!(\chi m-\chi)!}\cdot\frac{(\frac{t+\varepsilon}{2}+\chi m-1)!}{(\chi m-\chi)!(\frac{t+\varepsilon}{2}+\chi-1)!} \\ &= \frac{1}{(\chi m-\chi+1)\cdot\ldots\cdot(\chi m-1)}\binom{\chi m+[\frac{t}{2}]-\chi}{\chi m-\chi}\cdot\binom{\chi m+[\frac{t+1}{2}]-1}{\chi m-\chi}.\end{aligned}$$

The product in the denominator is 1 for $\chi=1$ and $2m-1$ for $\chi=2$. Thus, (2.21) and (2.22) are valid.□

2.2 Integration of zonal functions

Here we derive some integration formulas we have used in the main text. We denote by $\widetilde{\sigma}$ the Lebesgue measure (**area**) on the sphere $\mathbf{S}^{q-1} \equiv \mathbf{S}(\mathbf{R}^q)$ induced by the standard Lebesgue measure (volume) in $\mathbf{R}^q$. The **normalized measure** on $\mathbf{S}^{q-1}$ will be denoted by σ, so that

$$\sigma = \frac{\widetilde{\sigma}}{\mathrm{Area}(\mathbf{S}^{q-1})}. \tag{2.25}$$

From now on for any measure μ we use the short notation

$$\int f \, d\mu$$

meaning the integration over the support of μ or a set $Z \supset \mathrm{supp}\mu$.

THEOREM 2.2.1. *Let f be a continuous function on $[-1,1]$. Then for all $x \in \mathbf{S}^{q-1}$*

$$\boxed{\int f(\langle x, y\rangle)\, d\sigma(y) = \int_{-1}^{1} f(u)\Omega_q(u)\, du} \quad . \tag{2.26}$$

Proof. Consider the decomposition $\mathbf{R}^q = \mathrm{Span}(x) \oplus L$, so that $L \perp x$ and $y = \xi_1 x + z,\ z \in L$. Let σ' be the area on the unit sphere $\mathbf{S}(L) \equiv \mathbf{S}^{q-2}$ induced by $\widetilde{\sigma}$. Then

$$\boxed{d\widetilde{\sigma}(y) = (1-\xi_1^2)^{\frac{q-3}{2}} d\xi_1 d\sigma'(\widehat{z})} \quad . \tag{2.27}$$

(Recall that $\widehat{z} = z/\|z\|$ so, $\widehat{z} \in \mathbf{S}(L)$.) As a result,

$$\int f(\langle x, y\rangle)\, d\sigma(y) = \kappa \int_{-1}^{1} f(\xi_1)(1-\xi_1^2)^{\frac{q-1}{2}}\, d\xi_1 = \kappa' \int_{-1}^{1} f(u)\Omega_m(u)\, du \tag{2.28}$$

where κ and κ' are some coefficients. Actually, $\kappa' = 1$ since the measure σ and the weight Ω_m are both normalized. □

Now we obtain a modification of (2.26) regarding to the projective situation. The latter means that the integrand only depends on $|\langle x, y\rangle|$ or, equivalently, on $|\langle x, y\rangle|^2$. We start with a multi-dimensional counterpart of (2.27).

LEMMA 2.2.2. *Let $2 \le l \le q-2$. The measure $\widetilde{\sigma}$ is the product*

$$\boxed{d\widetilde{\sigma}(y) = (1-\rho^2)^{\frac{l}{2}-1}\rho^{q-l-1} d\rho d\widetilde{\sigma}_{l-1}(\widehat{z}) d\widetilde{\sigma}_{q-l-1}(\widehat{w})} \tag{2.29}$$

where $y = [\zeta_i]_1^q \in \mathbf{S}^{q-1}$, $z = [\zeta_i]_1^l$, $w = [\zeta_i]_{l+1}^q$, $\varrho = \|w\|$ and $\widetilde{\sigma}_{i-1}$ is the measure (area) induced on the sphere $\mathbf{S}^{i-1} \subset \mathbf{R}^i$, $2 \le i \le q-1$.

Proof. There is the diffeomorphism

$$y \mapsto (\rho, \widehat{z}, \widehat{w}), \quad 0 < \rho < 1, \quad \widehat{z} \in \mathbf{S}^{l-1}, \quad \widehat{w} \in \mathbf{S}^{q-l-1}, \tag{2.30}$$

its inverse diffeomorphism is

$$y = \begin{bmatrix} \sqrt{1-\rho^2}\widehat{z} \\ \rho\widehat{w} \end{bmatrix}. \tag{2.31}$$

Denote by $\vartheta_1, \dots, \vartheta_{l-1}$ and $\varphi_1, \dots, \varphi_{q-l-1}$ the spherical coordinates on $\mathbf{S}^{l-1}$ and $\mathbf{S}^{q-l-1}$ respectively, so that $(\zeta_1, \dots, \zeta_q) \mapsto (\rho, \vartheta_1, \dots, \vartheta_{l-1}, \varphi_1, \dots, \varphi_{q-l-1})$ instead of (2.30). The corresponding Jacobi matrix is

$$\begin{bmatrix} -\frac{\rho}{\sqrt{1-\rho^2}}\widehat{z} & \sqrt{1-\rho^2}Z & 0 \\ \widehat{w} & 0 & \rho W \end{bmatrix}, \tag{2.32}$$

where

$$Z = [\frac{\partial\widehat{\zeta}_i}{\partial\vartheta_k}], \ 1 \le i \le l, \ 1 \le k \le l-1,$$

and

$$W = [\frac{\partial\widehat{\zeta}_i}{\partial\varphi_k}], \ l+1 \le i \le q, \ 1 \le k \le q-l-1.$$

The first column in (2.32) is orthogonal to the others since

$$\sum_{i=1}^{l} \widehat{\zeta}_i \frac{\partial\widehat{\zeta}_i}{\partial\vartheta_k} = \frac{1}{2}\frac{\partial}{\partial\vartheta_k}\left(\sum_{i=1}^{l}\widehat{\zeta}_i^2\right) = 0$$

and, similarly,

$$\sum_{i=l+1}^{q} \widehat{\zeta}_i \frac{\partial\widehat{\zeta}_i}{\partial\varphi_k} = 0.$$

Since the norm of the first column is

$$\sqrt{\frac{\rho^2}{1-\rho^2} + 1} = (1-\rho^2)^{-\frac{1}{2}},$$

the corresponding Gram matrix is

$$G = \begin{bmatrix} (1-\rho^2)^{-1} & 0 & 0 \\ 0 & (1-\rho^2)Z'Z & 0 \\ 0 & 0 & \rho^2 W'W \end{bmatrix}.$$

However, Z and W are the Jacobi matrices of the transformations $(\vartheta_1, \dots, \vartheta_{l-1}) \mapsto (\widehat{\zeta}_1, \dots, \widehat{\zeta}_l)$ and $(\varphi_1, \dots, \varphi_{q-l-1}) \mapsto (\widehat{\zeta}_{l+1}, \dots, \widehat{\zeta}_q)$ respectively. Therefore,

$$d\widetilde{\sigma}(y) = \sqrt{\det G}\, d\rho\vartheta_1 \dots d\vartheta_{l-1} d\varphi_1 \dots d\varphi_{q-l-1} = (1-\rho^2)^{\frac{l}{2}-1}\rho^{q-l-1} d\rho d\widetilde{\sigma}_{l-1}(\widehat{z}) d\widetilde{\sigma}_{q-l-1}(\widehat{w}).$$

□

REMARK 2.2.3. Formula (2.29) is also valid for $l = 1$. *The measure* $\widetilde{\sigma}_0$ *on the* 0*-dimensional unit sphere* $\mathbf{S}^0 = \{-1, 1\} \subset \mathbf{R}$ *is such that* $\widetilde{\sigma}_0(1) = \widetilde{\sigma}_0(-1) = 1$. $\Box$

Below we apply Lemma 2.2.2 to $l = \delta$, $q = \delta m$ with $\delta = [\mathbf{K} : \mathbf{R}]$ and $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Then $\mathbf{S}^{q-1} = S(E_{\mathbf{R}}) = \mathbf{S}(E)$ where E is a m-dimensional ($m \geq 2$) right linear Euclidean space over $\mathbf{K}$ and $E_{\mathbf{R}}$ is the realification of E.

THEOREM 2.2.4. *Let* Φ *be a continuous function on* $[0, 1]$. *Then for all* $x \in \mathbf{S}(E)$

$$\boxed{\int \Phi(|\langle x, y\rangle|^2)\, d\sigma(y) = \int_{-1}^{1} \Phi\left(\frac{1+v}{2}\right) \Omega_{\alpha,\beta}(v)\, dv} \quad , \tag{2.33}$$

where

$$\boxed{\alpha = \frac{\delta m - \delta - 2}{2}, \quad \beta = \frac{\delta - 2}{2}} \quad . \tag{2.34}$$

Proof. Consider the coordinate system in $E \equiv \mathbf{K}^m$ with the first basis vector $x \in \mathbf{S}(E)$. If $y = [\zeta_i]_1^{\delta m}$ then $\langle x, y\rangle = \zeta_1 \in \mathbf{K} \equiv \mathbf{R}^\delta$ so, $z = \zeta_1$, $w = [\zeta_i]_2^m \in \mathbf{K}^{m-1} \equiv \mathbf{R}^{\delta m - 1}$ in notation of Lemma 2.2.2. Applying this lemma for $\delta = 2, 4$ and Remark 2.2.3 for $\delta = 1$ we obtain

$$\begin{aligned} \int \Phi(|\langle x, y\rangle|^2)\, d\sigma(y) &= \int \Phi(|\zeta_1|^2)\, d\sigma(y) \\ &= \kappa \int d\widetilde{\sigma}_{\delta-1} \int d\widetilde{\sigma}_{\delta m - \delta - 1} \int_0^1 \Phi(1-\rho^2)(1-\rho^2)^{\frac{\delta}{2}-1} \rho^{\delta m - \delta - 1}\, d\rho \\ &= \kappa_1 \int_0^1 \Phi(1-\rho^2) \rho^{2\alpha+1}(1-\rho^2)^\beta\, d\rho \end{aligned} \tag{2.35}$$

where κ and κ_1 are some coefficients. By substitution $1 - \rho^2 = \frac{1}{2}(1+v)$,

$$\int \Phi(|\langle x, y\rangle|^2)\, d\sigma(y) = \kappa_2 \int_{-1}^{1} \Phi\left(\frac{1+v}{2}\right)(1-v)^\alpha(1+v)^\beta\, dv = \kappa_3 \int_{-1}^{1} \Phi\left(\frac{1+v}{2}\right) \Omega_{\alpha,\beta}(v)\, dv \tag{2.36}$$

with some coefficients κ_2 and κ_3. In fact, we get $\kappa_3 = 1$ taking $\Phi = \mathbf{1}$ as before. $\Box$

COROLLARY 2.2.5. *For* $t \in \mathbf{N}$ *the quantity*

$$\boxed{\Upsilon_{\mathbf{K}}(m, t) = \left(\int |\langle x, y\rangle|^{2t}\, d\sigma(y)\right)^{-1}, \quad x \in \mathbf{S}(E)} \quad . \tag{2.37}$$

is independent of x, *namely,*

$$\boxed{\Upsilon_{\mathbf{R}}(m, t) = \frac{(2t + m - 2)!!}{(m-2)!!(2t-1)!!}} \tag{2.38}$$

and

$$\boxed{\Upsilon_{\mathbf{C}}(m,t) = \binom{t+m-1}{m-1}} \quad , \tag{2.39}$$

and

$$\boxed{\Upsilon_{\mathbf{H}}(m,t) = \frac{1}{t+1}\binom{t+2m-1}{2m-1}} \quad . \tag{2.40}$$

Proof. By Theorem 2.2.4

$$\int |\langle x,y\rangle|^{2t}\, d\sigma(y) = \int_{-1}^{1} \left(\frac{1+v}{2}\right)^t \Omega_{\alpha,\beta}(v)\, dv = \frac{1}{2^t\tau_{\alpha,\beta}} \int_{-1}^{1} \omega_{\alpha,\beta+t}(v)\, dv = \frac{\tau_{\alpha,\beta+t}}{2^t\tau_{\alpha,\beta}}. \tag{2.41}$$

and by substitution from (2.17)

$$\Upsilon_{\mathbf{K}}(m,t) = \frac{2^t\tau_{\alpha,\beta}}{\tau_{\alpha,\beta+t}} = \frac{\Gamma(\beta+1)\Gamma(t+\alpha+\beta+2)}{\Gamma(\alpha+\beta+2)\Gamma(t+\beta+1)}, \tag{2.42}$$

i.e.

$$\boxed{\Upsilon_{\mathbf{K}}(m,t) = \frac{\Gamma(\frac{\delta}{2})\Gamma(t+\frac{\delta}{2}m)}{\Gamma(\frac{\delta}{2}m)\Gamma(t+\frac{\delta}{2})}} \quad . \tag{2.43}$$

If $\mathbf{K} = \mathbf{R}$, i.e. $\delta = 1$, then by (2.24)

$$\Upsilon_{\mathbf{R}}(m,t) = \frac{\Gamma(\frac{1}{2})\Gamma(t+\frac{m}{2})}{\Gamma(t+\frac{1}{2})\Gamma(\frac{m}{2})} = \frac{(m+2t-2)!!}{(m-2)!!(2t-1)!!}. \tag{2.44}$$

If $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$, i.e. $\delta = 2$ or 4, then

$$\Upsilon_{\mathbf{K}}(m,t) = \frac{(t+\frac{\delta}{2}m-1)!}{(\frac{\delta}{2}m-1)!(t+\frac{\delta}{2}-1)!} = \frac{t!}{(t+\frac{\delta}{2}-1)!}\binom{t+\frac{\delta}{2}m-1}{\frac{\delta}{2}m-1}. \tag{2.45}$$

The latter fraction in (2.45) is equal to 1 or $\frac{1}{t+1}$ if $\delta = 2$ or 4 respectively. ☐

Note that

$$\boxed{\Upsilon_{\mathbf{K}}(m,0) = 1, \quad \Upsilon_{\mathbf{K}}(m,1) = m} \tag{2.46}$$

irrespective to **K**.

COROLLARY 2.2.6. *For all $x \in E$* **the Hilbert Identity**

$$\boxed{\langle x,x\rangle^t = \Upsilon_{\mathbf{K}}(m,t) \int |\langle x,y\rangle|^{2t}\, d\sigma(y)} \tag{2.47}$$

holds.

2.3 Some differential operators

Here we derive some partial differential equations for the functions

$$u_y(x) = |\langle x, y\rangle|, \quad z_y(x) = \langle x, y\rangle, \quad w_y(x) = \frac{\langle x, y\rangle}{\|x\|}. \tag{2.48}$$

with a fixed vector $y \in E$. Note that the restrictions of functions (2.48) to $\mathbf{S}(E)$ are polynomial functions on $\mathbf{S}(E)$.

First of all, we have

LEMMA 2.3.1. *For $x = [\xi_i]_1^m \in \mathbf{R}^m$ the equations*

$$\sum_{i=1}^{m} \left(\frac{\partial w_y(x)}{\partial \xi_i}\right)^2 = \frac{1}{\|x\|^2}(\|y\|^2 - w_y^2(x)) \tag{2.49}$$

and

$$\Delta w_y(x) = (1-m)\frac{w_y(x)}{\|x\|^2} \tag{2.50}$$

are valid.

Proof. Let $y = [\zeta_i]_1^m \in \mathbf{R}^m$. Then

$$\frac{\partial w_y}{\partial \xi_i} = \frac{\zeta_i \|x\|^2 - z_y \xi_i}{\|x\|^3}, 1 \le i \le m. \tag{2.51}$$

Using (2.51) we get (2.49)

$$\sum_{i=1}^{m} \left(\frac{\partial w_y(x)}{\partial \xi_i}\right)^2 = \sum_{i=1}^{m} \left(\frac{\zeta_i^2}{\|x\|^2} + \frac{z_y^2 \xi_i^2}{\|x\|^6} - \frac{2\zeta_i \xi_i z_y}{\|x\|^4}\right) = \frac{1}{\|x\|^2}(\|y\|^2 - w_y^2(x)).$$

Similarly,

$$\begin{aligned} \Delta w_y &= \sum_{i=1}^{m} \frac{\partial^2 w_y}{\partial \xi_i^2} = \frac{1}{\|x\|^6} \sum_{i=1}^{m} \left(2\zeta_i \xi_i \|x\|^3 - \zeta_i \xi_i \|x\|^3 - z_y \|x\|^3 - 3\|x\|\xi_i(\zeta_i \|x\|^2 - z_y \xi_i)\right) \\ &= (1-m)\frac{z_y}{\|x\|^3} = (1-m)\frac{w_y}{\|x\|^2} \end{aligned}$$

which is (2.50). □

The next statement is

LEMMA 2.3.2. *Let $f : [-1, 1] \to \mathbf{C}$ be a twice differentiable function and let*

$$\widetilde{f}(x) = \|x\|^k f(w) \tag{2.52}$$

where

$$w = w_y(x), \quad x \in E, \quad y \in \mathbf{S}(E). \tag{2.53}$$

Then

$$\Delta \widetilde{f}_k(x) = \|x\|^{k-2}\Big(k(k+m-2)f(w) + (1-m)wf'(w) + (1-w^2)f''(w)\Big). \tag{2.54}$$

Proof. Obviously,

$$\frac{\partial \widetilde{f}}{\partial \xi_i} = k\|x\|^{k-2}\xi_i f(w) + \|x\|^k \frac{\partial w}{\partial \xi_i} f'(w)$$

whence,

$$\begin{aligned} \frac{\partial^2 \widetilde{f}}{\partial \xi_i^2} &= k(k-2)\|x\|^{k-4}\xi_i^2 f(w) + k\|x\|^{k-2} f(w) + 2k\|x\|^{k-2}\xi_i \frac{\partial w}{\partial \xi_i} f'(v_y) \\ &+ \|x\|^k \frac{\partial^2 w}{\partial \xi_i} f'(w) + \|x\|^k \left(\frac{\partial w}{\partial \xi_i}\right)^2 f''(w). \end{aligned}$$

It remains to apply Lemma 2.3.1. □

Further we use the standard differential operators

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - \mathbf{i}\frac{\partial}{\partial \beta}\right), \quad \frac{\partial}{\partial \overline{\xi}} = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} + \mathbf{i}\frac{\partial}{\partial \beta}\right) \tag{2.55}$$

with respect to a complex variable $\xi = \alpha + \beta\mathbf{i}$. Obviously, for any real-valued function z

$$\frac{\partial z}{\partial \overline{\xi}} = \overline{\left(\frac{\partial z}{\partial \xi}\right)}. \tag{2.56}$$

For the functions $\mathbf{C}^m \to \mathbf{C}$ we consider the standard Laplacian written as

$$\Delta = 4\sum_{i=1}^{m} \frac{\partial^2}{\partial \xi_i \partial \overline{\xi_i}} \tag{2.57}$$

where ξ_i, $1 \le i \le m$, are the canonical coordinates. Similarly, according to (1.1),

$$\Delta = 4\sum_{i=1}^{m} \left(\frac{\partial^2}{\partial \xi_i \partial \overline{\xi_i}} + \frac{\partial^2}{\partial \eta_i \partial \overline{\eta_i}}\right) \tag{2.58}$$

in the quaternionic case.

LEMMA 2.3.3. *For* $x = [\xi_i + \eta_i\mathbf{j}]_1^m \in \mathbf{H}^m$ *the equations*

$$\sum_{i=1}^{m} \left(\frac{\partial u_y(x)}{\partial \xi_i} \cdot \frac{\partial u_y(x)}{\partial \overline{\xi_i}} + \frac{\partial u_y(x)}{\partial \eta_i} \cdot \frac{\partial u_y(x)}{\partial \overline{\eta_i}}\right) = \frac{1}{4}\|y\|^2 \tag{2.59}$$

and

$$u_y(x)\Delta u_y(x) = 3\|y\|^2 \tag{2.60}$$

hold.

Proof. Since both sides of the equations (2.59) and (2.60) are continuous, the calculations can be done under restriction $u_y(x) \neq 0$.

Let $y = [\zeta_i + \omega_i\mathbf{j}]_1^m \in \mathbf{H}^m$. According to (1.62) and (1.61) we have

$$u_y^2(x) = |\langle x, y\rangle|^2 = \sum_{s,l=1}^{m} (a_s\overline{a}_l + b_s\overline{b}_l) \tag{2.61}$$

where

$$a_i = \overline{\xi}_i\zeta_i + \eta_i\overline{\omega}_i, \quad b_i = \overline{\xi}_i\omega_i - \eta_i\overline{\zeta}_i. \tag{2.62}$$

Therefore,

$$\frac{\partial u_y}{\partial \xi_i} = \frac{1}{2u_y}\frac{\partial u_y^2}{\partial \xi_i} = \frac{1}{2u_y}\left(\frac{\partial a_i}{\partial \xi_i}\sum_{s=1}^m \overline{a}_s + \frac{\partial \overline{a}_i}{\partial \xi_i}\sum_{s=1}^m a_s + \frac{\partial b_i}{\partial \xi_i}\sum_{s=1}^m \overline{b}_s + \frac{\partial \overline{b}_i}{\partial \xi_i}\sum_{s=1}^m b_s\right)$$

or

$$\frac{\partial u_y}{\partial \xi_i} = \frac{1}{2u_y}\left(\overline{\zeta_i}\sum_{s=1}^m a_s + \overline{\omega_i}\sum_{s=1}^m b_s\right) \tag{2.63}$$

Similarly,

$$\frac{\partial u_y}{\partial \eta_i} = \frac{1}{2u_y}\left(\overline{\omega_i}\sum_{s=1}^m \overline{a}_s - \overline{\zeta_i}\sum_{s=1}^m \overline{b}_s\right). \tag{2.64}$$

Now (2.63) and (2.64) imply

$$\frac{\partial u_y}{\partial \xi_i}\cdot\frac{\partial u_y}{\partial \overline{\xi}_i} = \frac{1}{4u_y^2}\left(\zeta_i\overline{\zeta}_i\sum_{s,l=1}^m a_s\overline{a}_l + \omega_i\overline{\omega} i\sum_{s,l=1}^m b_s\overline{b}_l + \overline{\zeta}_i\omega_i\sum_{s,l=1}^m a_s\overline{b}_l + \zeta_i\overline{\omega}_i\sum_{s,l=1}^m \overline{a}_s b_l\right) \tag{2.65}$$

and

$$\frac{\partial u_y}{\partial \eta_i}\cdot\frac{\partial u_y}{\partial \overline{\eta}_i} = \frac{1}{4u_y^2}\left(\omega_i\overline{\omega}_i\sum_{s,l=1}^m a_s\overline{a}_l + \zeta_i\overline{\zeta}_i\sum_{s,l=1}^m b_s\overline{b}_l - \zeta_i\overline{\omega}_i\sum_{s,l=1}^m \overline{a}_s b_l - \overline{\zeta}_i\omega_i\sum_{s,l=1}^m a_s\overline{\beta}_l\right).$$

Therefore, by (2.61)

$$\sum_{i=1}^m\left(\frac{\partial u_y}{\partial \xi_i}\cdot\frac{\partial u_y}{\partial \overline{\xi_i}} + \frac{\partial u_y}{\partial \eta_i}\cdot\frac{\partial u_y}{\partial \overline{\eta_i}}\right) = \frac{1}{4u_y^2}\sum_{i=1}^m(\zeta_i\overline{\zeta}_i + \omega_i\overline{\omega}_i)\sum_{s,l=1}^m(a_s\overline{a}_l + b_s\overline{b}_l) = \frac{1}{4}\|y\|^2,$$

so, we have got (2.59).

Finally, it follows from (2.63) that

$$\frac{\partial^2 u_y}{\partial \xi_i\partial \overline{\xi}_i} = \frac{\partial u_y}{\partial \overline{\xi}_i}\left(\overline{\zeta_i}\sum_{s=1}^m a_s + \overline{\omega_i}\sum_{s=1}^m b_s\right) = \frac{1}{2u_y^2}\left(u_y\Big(\zeta_i\overline{\zeta}_i + \eta_i\overline{\eta}_i\Big) - (\overline{\zeta_i}\sum_{s=1}^m a_s + \overline{\omega_i}\sum_{s=1}^m b_s)\frac{\partial u_y}{\partial \overline{\xi}_i}\right)$$

by (2.62). Using (2.63) again we get

$$\frac{\partial^2 u_y}{\partial \xi_i\partial \overline{\xi}_i} = \frac{1}{2u_y}\left(\zeta_i\overline{\zeta}_i + \eta_i\overline{\eta}_i - 2\frac{\partial u_y}{\partial \xi_i}\cdot\frac{\partial u_y}{\partial \overline{\xi}_i}\right). \tag{2.66}$$

Likewise,

$$\frac{\partial^2 u_y}{\partial \eta_i\partial \overline{\eta}_i} = \frac{1}{2u_y}\left(\zeta_i\overline{\zeta}_i + \eta_i\overline{\eta}_i - 2\frac{\partial u_y}{\partial \eta_i}\cdot\frac{\partial u_y}{\partial \overline{\eta}_i}\right).$$

By (2.59)

$$u_y\Delta u_y = 4\sum_{i=1}^m(\zeta_i\overline{\zeta}_i + \eta_i\overline{\eta}_i) - 4\sum_{i=1}^m\left(\frac{\partial u_y}{\partial \xi_i}\cdot\frac{\partial u_y}{\partial \overline{\xi_i}} + \frac{\partial u_y}{\partial \eta_i}\cdot\frac{\partial u_y}{\partial \overline{\eta_i}}\right) = 4\|y\|^2 - \|y\|^2 = 3\|y\|^2.$$

□

The complex version of Lemma 2.3.3 is

LEMMA 2.3.4. *For $x = [\xi_i]_1^m \in \mathbf{C}^m$ the equations*

$$\sum_{i=1}^{m} \frac{\partial u_y(x)}{\partial \xi_i} \cdot \frac{\partial u_y(x)}{\partial \overline{\xi_i}} = \frac{1}{4}\|y\|^2, \tag{2.67}$$

and

$$u_y(x)\Delta u_y(x) = \|y\|^2 \tag{2.68}$$

hold.

Proof. By the canonical inclusion $\mathbf{C}^m \subset \mathbf{H}^m$ we can extend the function $u_y(x)$ with $y = [\zeta_i]_1^m \subset \mathbf{C}^m$ to the quaternion space. Thus, $u_y = \mathcal{U}|\mathbf{C}^m$ where $\mathcal{U}(x) = |\langle x, y\rangle|$, $x \in \mathbf{H}^m$. Then

$$\frac{\partial u_y}{\partial \xi_i} = \frac{\partial \mathcal{U}}{\partial \xi_i}\Big|_{\mathbf{C}^m}, \quad 1 \le i \le m. \tag{2.69}$$

Also note that for $y = [\zeta_i + \omega_i \mathbf{j}]_1^m \in \mathbf{H}^m$ we have $\omega_i = 0,\ 1 \le i \le m$, so that $b_i = 0$ if $x \in \mathbf{C}^m$. As a result,

$$\frac{\partial u_y}{\partial \eta_i}\Big|_{\mathbf{C}^m} = 0, \quad 1 \le i \le m, \tag{2.70}$$

see (2.64).

Taking into account (2.69) and (2.70) we get (2.67) from (2.59). Now (2.68) follows from (2.66) and (2.67) with $\eta_i = 0$, $1 \le i \le m$. □

As a consequence of Lemmas 2.3.3 and 2.3.4 we obtain the most important for our purposes

LEMMA 2.3.5. *The equation*

$$\boxed{\Delta u_y^{2k}(x) = 2k(2k + \delta - 2)\|y\|^2 u_y^{2k-2}(x)} \tag{2.71}$$

holds for $x \in \mathbf{K}^m$, $\delta = [\mathbf{K} : \mathbf{R}]$, i.e. $\delta = 1, 2, 4$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ respectively.

Proof. In our notation $|\langle x, y\rangle|^{2k} = u_y^{2k}(x)$. Further we consider three separate cases.

1. $\mathbf{K} = \mathbf{H}$. A direct calculation yields

$$\Delta u_y^{2k} = 8k(2k-1)u_y^{2k-2} \sum_{i=1}^{m} \left(\frac{\partial u_y}{\partial \xi_i} \cdot \frac{\partial u_y}{\partial \overline{\xi_i}} + \frac{\partial u_y}{\partial \eta_i} \cdot \frac{\partial u_y}{\partial \overline{\eta_i}} \right) + 2k u_y^{2k-1} \Delta u_y.$$

By Lemma 2.3.3.

$$\Delta u_y^{2k} = 2k(2k-1)\|y\|^2 u_y^{2k-2} + 6k\|y\|^2 u_y^{2k-2} = 2k(2k+2)\|y\|^2 u_y^{2k-2}.$$

2. $\mathbf{K} = \mathbf{C}$. As before

$$\Delta u_y^{2k} = 2k(2k-1)u_y^{2k-2} \sum_{i=1}^{m} \frac{\partial u_y}{\partial \xi_i} \cdot \frac{\partial u_y}{\partial \overline{\xi_i}} + 2k u_y^{2k-1} \Delta u_y = 4k^2 \|y\|^2 u_y^{p-2}$$

by Lemma 2.3.4.

3. $\mathbf{K} = \mathbf{R}$ We have

$$\Delta u_y^{2k} = \Delta z_y^{2k} = (2k-1)(z_y(x))^{2k-2} \sum_{i=1}^{m} \left(\frac{\partial z_y}{\partial \xi_i} \right)^2 + 2k z_y^{2k-1} \Delta z_y = 2k(2k-1) z_y^{2k-2}$$

since

$$z_y(x) = \sum_{k=1}^{m} \xi_i \zeta_y.$$

□

Chapter 3

Polynomial functions

In this Chapter E is supposed to be a m-dimensional ($m \geq 2$) right linear Euclidean space over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, or $\mathbf{H}$. We will deal with complex-valued polynomial functions on the real unit sphere $\mathbf{S}(E)$ and the projective space $\mathbf{P}(E)$. The most fundamental is the case of polynomial functions on $\mathbf{S}(E)$.

All functional spaces under consideration will be considered as complex linear spaces

3.1 Polynomial functions on the real unit spheres

Within this Section E is a real m-dimensional Euclidean space.

The unit sphere $\mathbf{S}(E)$ is a real algebraic manifold. In spirit of Algebraic Geometry we define a **polynomial function** $\phi : \mathbf{S}(E) \to \mathbf{C}$ as a restriction to $\mathbf{S}(E)$ of a polynomial $\psi : E \to \mathbf{C}$.

Given a basis in E, the general form of **polynomials** on E is

$$\psi(x) = \sum_I \alpha_I [\xi]^I \tag{3.1}$$

where I runs over a finite set of multiindices $(i_1, \ldots, i_m) \in \mathbf{N}^m$, α_I are complex coefficients and

$$[\xi]^I = \xi_1^{i_1} \cdots \xi_m^{i_m}$$

are the corresponding monomes with respect to the coordinates $\xi_1, \ldots, \xi_m$ of x. If

$$|I| = \sum_{k=1}^{m} i_k \tag{3.2}$$

then $\deg \psi$, the **degree** of ψ, is the maximal value of $|I|$ in (3.1) with constraint $\alpha_I \neq 0$. This number is independent of the choice of basis since for given ψ the coefficients α_I are uniquely determined. At least one of them is different from zero if $\psi(x)$ is not identically zero ($\psi = 0$ is the only case when $\deg \psi$ is not defined because all $\alpha_I = 0$). The set $\mathcal{P}(E)$ of all polynomials on E is a (infinite-dimensional) linear space with respect to the standard linear operations in functional spaces. (Moreover, $\mathcal{P}(E)$ is a ring so that $\mathcal{P}(E)$ is an algebra over $\mathbf{C}$.) Hence, the set $\texttt{Pol}(E)$ of all polynomial functions on $\mathbf{S}(E)$ is also a linear space, the image of $\mathcal{P}(E)$ under the **restriction homomorphism** $r : \mathcal{P}(E) \to \texttt{Pol}(E)$ defined as

$$r\psi = \psi|\mathbf{S}(E) \tag{3.3}$$

There is a lot of polynomials $\psi \in \mathcal{P}(E)$ which generate the same polynomial function $\phi \in \texttt{Pol}(E)$ by (3.3). The point is that the kernel of the homomorphism r is the (infinite-dimensional) subspace

$$\mathcal{P}^0(E) = \{\psi \in \mathcal{P}(E) : \psi(x) = (1 - \|x\|^2)\omega(x),\ \omega \in \mathcal{P}(E). \tag{3.4}$$

Indeed, let $\theta(x) = 1 - \|x\|^2 = 1 - \langle x, x\rangle$. If $\theta(x) = 0$ then θ is a divisor of a power ψ^ν by Hilbert's Nullstellensatz. Since θ is absolutely irreducible, θ is a divisor of ψ, i.e. $\psi = \theta\omega$.

For any $\phi \in \texttt{Pol}(E) \setminus \{0\}$ we define its **degree** as the minimum of degrees of $\psi \in \mathcal{P}(E)$ such that $\phi = r\psi$.

Note that a definite lifting $\phi \to \psi$ does exist if the polynomial ψ is homogeneous,

$$\psi(x\gamma) = \gamma^d\psi(x), \quad \gamma \in \mathbf{R}, \tag{3.5}$$

where $d = \deg\psi$ (in other words ψ is a **form** of degree d). In this case (3.3) implies

$$\psi(x) = \|x\|^d\phi(\widehat{x}), \quad x \in E. \tag{3.6}$$

Formula (3.6) defines that we call the **homogeneous lifting**.

In the nonhomogeneous case

$$\psi(x) = \sum_{k=0}^{d}\psi_k(x), \quad x \in E, \tag{3.7}$$

where $d = \deg\psi$ as before, ψ_k is the homogeneous component of degree k (or $\psi_k = 0$). By restriction of (3.7) we get

$$\phi(x) = \sum_{k=0}^{d}\phi_k(x), \quad x \in \mathbf{S}(E), \tag{3.8}$$

where $\phi_k = \psi_k|\mathbf{S}(E)$, $0 \le k \le d$. Then

$$\psi(x) = \sum_{k=0}^{d}\|x\|^k\phi_k(x), \quad x \in E. \tag{3.9}$$

In this way we recover the polynomial ψ as soon as all homogeneous components of ψ are given. However, for a given polynomial function ϕ the decomposition (3.8) into the sum of restrictions of forms of degrees $k = 0, 1, \ldots, d$ is not unique in view of (3.4). In order to overcome this difficulty we have to restrict the space $\mathcal{P}(E)$ to its subset $\mathcal{H}(E)$ consisting of harmonic polynomials on E,

$$\mathcal{H}(E) = \{\psi \in \mathcal{P}(E) :\ \Delta\psi = 0\}. \tag{3.10}$$

The subset $\mathcal{H}(E)$ is a (infinite-dimensional) subspace of $\mathcal{P}(E)$, the kernel of the Laplacian

$$\Delta = \sum_{i=1}^{m}\frac{\partial^2}{\partial\xi_i^2} \tag{3.11}$$

which is a linear operator $\mathcal{P}(E) \to \mathcal{P}(E)$. An important property of this operator is the **orthogonal invariance**,

$$\boxed{\Delta(\psi(gx)) = (\Delta\psi)(gx), \quad g \in O(E)} \quad . \tag{3.12}$$

The classical uniqueness theorem for the Dirichlet problem says that *there are no harmonic functions vanishes on the unit sphere* $\mathbf{S}(E)$. This immediately yields

LEMMA 3.1.1. *The subspaces* $\mathcal{P}^0(E)$ *and* $\mathcal{H}(E)$ *are independent, i.e.*

$$\boxed{\mathcal{P}^0(E) \cap \mathcal{H}(E) = 0} \quad . \tag{3.13}$$

Let us also introduce

$$\mathcal{P}_d(E) = \{\psi \in \mathcal{P}(E) : \deg\psi \le d \text{ or } \psi = 0\}, \tag{3.14}$$

a finite-dimensional subspace of $\mathcal{P}(E)$, and then

$$\mathcal{H}_d(E) = \mathcal{P}_d(E) \cap \mathcal{H}(E), \tag{3.15}$$

the subspace of harmonic polynomials of degree $\le d$.

By restriction of r we obtain the corresponding subspaces of $\mathtt{Pol}(E)$, namely,

$$\mathtt{Harm}(E) = r\mathcal{H}(E) \tag{3.16}$$

and

$$\mathtt{Harm}_d(E) = r\mathcal{H}_d(E), \;\; \mathtt{Pol}_d(E) = r\mathcal{P}_d(E). \tag{3.17}$$

Thus, $\mathtt{Pol}_d(E)$ is the space of all polynomial functions of degrees $\le d$, $\mathtt{Harm}_d(E)$ is a subspace of $\mathtt{Pol}_d(E)$.

In addition, let

$$\mathcal{P}^0_d(E) = \mathcal{P}^0(E) \cap \mathcal{P}_d(E), \tag{3.18}$$

the subspace of those polynomials of degrees $\le d$ which vanish on $\mathbf{S}(E)$.

THEOREM 3.1.2. *The equality*

$$\boxed{\mathtt{Pol}_d(E) = \mathtt{Harm}_d(E)} \tag{3.19}$$

holds.

In other words, *any polynomial function of degree* $\leq d$ *can be obtained by restriction to* $\mathbf{S}(E)$ *of a harmonic polynomial of degree* $\leq d$. The latter is unique (even without the constraint for the degree) because of Lemma 3.1.1. This means that $r|\mathcal{H}(E)$ is an isomorphism from $\mathcal{H}(E)$ onto $\mathtt{Harm}(E)$, the **restriction isomorphism**. The same is also true for $r|\mathcal{H}_d(E) : \mathcal{H}_d(E) \to \mathtt{Harm}_d(E)$. As a result

$$\boxed{\mathtt{Pol}(E) = \mathtt{Harm}(E) \approx \mathcal{H}(E), \quad \mathtt{Harm}_d(E) \approx \mathcal{H}_d(E)} \ . \tag{3.20}$$

In order to prove Theorem 3.1.2 we need some statements which are interesting per se.

Consider the homomorphism $A_d^0 : \mathcal{P}_{d-2}(E) \to \mathcal{P}_d^0(E)$ defined as

$$(A_d^0\omega)(x) = (1 - \|x\|^2)\omega(x).$$

Obviously, A_d^0 is injective since there is no divisors of zero in the ring $\mathcal{P}(E)$. In addition, A_d^0 is surjective by (3.4). Hence, A_d^0 is bijective, so that we have the isomorphism, $\mathcal{P}_{d-2}(E) \approx \mathcal{P}_d^0(E)$. Therefore,

$$\boxed{\dim \mathcal{P}_d^0(E) = \dim \mathcal{P}_{d-2}(E)} \ . \tag{3.21}$$

The Laplacian Δ on $\mathcal{P}_d(E)$ is a homomorphism $\Delta_d : \mathcal{P}_d(E) \to \mathcal{P}_{d-2}(E)$ with $\mathrm{Ker}\Delta_d = \mathcal{H}_d(E)$. Let us denote $\Delta_d|\mathcal{P}_d^0(E)$ by Δ_d^0. This is a homomorphism $\mathcal{P}_d^0(E) \to \mathcal{P}_{d-2}(E)$ which is injective by Lemma 3.1.1. By (3.21) and Corollary 1.7.3 Δ_d^0 is bijective, i.e. it is an isomorphism. This result can be formulated as follows.

LEMMA 3.1.3. *For any polynomial* θ *of degree* $\leq d-2$ *the Dirichlet problem*

$$\Delta\psi = \theta, \quad \psi|\mathbf{S}(E) = 0$$

has a unique polynomial solution ψ, $\deg\psi \leq d$. *(If* $\theta = 0$ *then* $\psi = 0$.*)*

COROLLARY 3.1.4. *The direct decomposition*

$$\boxed{\mathcal{P}_d(E) = \mathcal{H}_d(E) \dot{+} \mathcal{P}_d^0(E)} \ . \tag{3.22}$$

holds.

Proof. First of all,

$$\mathcal{H}_d(E) \cap \mathcal{P}_d^0(E) = 0$$

by Lemma 3.1.1. Now let $\psi \in \mathcal{P}_d(E)$. Denote by $\psi^{(0)}$ the polynomial solution of degree $\leq d$ for the Dirichlet problem

$$\Delta\psi^{(0)} = \Delta\psi, \quad \psi^{(0)}|\mathbf{S}(E) = 0 \tag{3.23}$$

which does exist by Lemma 3.1.3. If $\psi^{(1)} = \psi - \psi^{(0)}$ then $\psi = \psi^{(1)} + \psi^{(0)}$ where $\psi^{(1)} \in \mathcal{H}_d(E)$ and $\psi^{(0)} \in \mathcal{P}_d^0(E)$ by (3.23). $\square$

The above mentioned polynomial $\psi^{(1)}$ is called the **harmonic projection of** ψ.

COROLLARY 3.1.5. $\mathcal{P}(E) = \mathcal{H}(E) \dot{+} \mathcal{P}^0(E)$.

In other words, *every polynomial* ψ *can be uniquely represented as*

$$\psi(x) = \psi^{(1)}(x) + (1 - \|x\|^2)\omega(x), \quad \omega \in \mathcal{P}(E). \tag{3.24}$$

where $\psi^{(1)}$ *is a harmonic polynomial.*

Now we can quickly prove Theorem 3.1.2.

Proof. Since $\mathtt{Harm}_d(E) \subset \mathtt{Pol}_d(E)$, we only need to prove the inverse inclusion. Let $\phi \in \mathtt{Pol}_d(E)$, $\phi = \psi|\mathbf{S}(E)$ where $\psi \in \mathcal{P}_d(E)$. By Corollary 3.1.4 $\psi = \psi^{(1)} + \psi^{(0)}$ where $\psi^{(1)} \in \mathcal{H}_d(E)$ and $\psi^{(0)} \in \mathcal{P}_d^0(E)$. Hence, $\phi = \psi^{(1)}|\mathbf{S}(E)$. □

The isomorphism r^{-1}, the inverse to $r : \mathcal{H}(E) \to \mathtt{Pol}(E)$ can be called the **harmonic lifting** for the polynomial functions. The harmonic lifting on $\mathcal{P}_d(E)$ maps this subspace onto $\mathcal{H}_d(E)$. `Later on the notation` $\phi^{(h)}$ `stands for the harmonic lifting of` $\phi \in \mathtt{Pol}(E)$`, so that` $\phi^{(h)} = r^{-1}\phi$.

Now we decompose the space $\mathcal{P}_d(E)$ into the direct sum

$$\boxed{\mathcal{P}_d(E) = \mathcal{P}(E; 0) \dot{+} \mathcal{P}(E; 1) \dot{+} \ldots \dot{+} \mathcal{P}(E; d)} \tag{3.25}$$

where $\mathcal{P}(E; k)$ is the subspace of all forms of degree k, $0 \le k \le d$. If, according to (3.25),

$$\psi = \sum_{k=0}^{d} \psi_k \tag{3.26}$$

(so that ψ_k are the homogeneous components of $\psi \in \mathcal{P}_d(E)$). Then

$$\Delta\psi = \sum_{k=0}^{d} \Delta\psi_k = \sum_{k=2}^{d} \Delta\psi_k \tag{3.27}$$

since $\Delta\psi_0 = 0$ (ψ_0 is a constant) and $\Delta\psi_1 = 0$ (ψ_1 is a linear function). In (3.27) we have the decomposition of $\Delta\psi \in \mathcal{P}_{d-2}(E)$ onto homogeneous components, since $\Delta\psi_k \in \mathcal{P}(E; k-2)$, $k \ge 2$. If $\Delta\psi = 0$ then all $\Delta\psi_k = 0$. Thus, we have proved

LEMMA 3.1.6. *For any harmonic polynomial* ψ *its homogeneous components are harmonic as well.*

As a consequence, the harmonic specialization of (3.25) is

$$\boxed{\mathcal{H}_d(E) = \mathcal{H}(E; 0) \dot{+} \mathcal{H}(E; 1) \dot{+} \ldots \dot{+} \mathcal{H}(E; d)} \tag{3.28}$$

where

$$\mathcal{H}(E; k) = \mathcal{P}(E; k) \cap \mathcal{H}(E), \tag{3.29}$$

the space of harmonic forms of degree k, $0 \le k \le d$ (cf. (3.15)). By restriction of these forms to the unit sphere $\mathbf{S}(E)$ we obtain the space $\mathtt{Harm}(E;k)$ of those polynomial functions which are just traditional **spherical harmonics of degree k.** As before, we have

$$\boxed{\mathtt{Harm}(E;k) \approx \mathcal{H}(E;k)} \tag{3.30}$$

by restriction $r : \mathcal{H}(E;k) \to \mathtt{Harm}(E;k)$. The corresponding harmonic lifting coincides with the homogeneous one.

The following result is classical (cf. [40]).

THEOREM 3.1.7. *The direct decomposition*

$$\boxed{\mathtt{Pol}_d(E) = \mathtt{Harm}(E;0)\dot{+}\mathtt{Harm}(E;1)\dot{+}\ldots\dot{+}\mathtt{Harm}(E;d)} \tag{3.31}$$

holds.

In other words, *any polynomial function ϕ on the unit sphere can be uniquely represented as a sum of spherical harmonic of pairwise distinct degrees.* They are called the **harmonic components** of ϕ. We call the decomposition (3.31) the **harmonic decomposition**. *The decomposition (3.28) is the harmonic lifting of (3.31).*

Proof. By Theorem 3.1.2 the decomposition (3.28) yields

$$\mathtt{Pol}_d(E) = \sum_{k=0}^{d} \mathtt{Harm}(E;k). \tag{3.32}$$

It remains to prove that the latter decomposition is direct. Suppose that

$$\sum_{k=0}^{d} \phi_k = 0 \tag{3.33}$$

where $\phi_k \in \mathtt{Harm}(E;k)$, i.e. $\phi_k = \psi_k|\mathbf{S}(E)$, where $\psi_k \in \mathcal{H}(E;k)$, $0 \le k \le d$. Then

$$\psi = \sum_{k=0}^{d} \psi_k \in \mathcal{H}(E) \tag{3.34}$$

and $\psi \in \mathcal{P}^0(E)$ by (3.33). By Lemma 3.1.1 $\psi = 0$. Since the decomposition (3.28) is direct, we get $\psi_k = 0,\ 0 \le k \le d$. Hence, all ϕ_k are zero. □

Consider some useful consequences of Theorem 3.1.7. Obviously, (3.31) implies

COROLLARY 3.1.8. *The direct decomposition*

$$\boxed{\mathtt{Pol}_d(E) = \mathtt{Pol}_{d-1}(E)\dot{+}\mathtt{Harm}(E;d)} \tag{3.35}$$

are valid.

Besides, we are able to prove the following

COROLLARY 3.1.9. *The degree of any $\phi \in \mathtt{Pol}(E) \setminus \{0\}$ coincides with the degree of its harmonic lifting $\phi^{(h)}$.*

Proof. By definition of the number $d = \deg \phi$, this is the minimum of $\deg \psi$ among all ψ in (3.24) where $\psi^{(1)} = \phi^{(h)}$. Taking $\omega = 0$ we get $d \leq \deg \phi^{(h)}$. Suppose that $d < \deg \phi^{(h)}$ and consider ψ of form (3.24) such that $\deg \psi = d$. Then the degree of the polynomial $(1 - \|x\|^2)\omega(x)$ is also d, so that $\deg \omega = d - 2$. Moreover, the d-th homogeneous component $\phi_d^{(h)}$ of $\phi^{(h)}$ is

$$\phi_d^{(h)}(x) = \|x\|^2 \omega_{d-2}(x), \tag{3.36}$$

where ω_{d-2} is the $(d-2)$-th homogeneous component of ω. The component $\phi_d^{(h)}$ is harmonic by Lemma 3.1.6. Therefore, $\omega_{d-2}|\mathbf{S}(E) \in \mathtt{Harm}(E; d)$. On the other hand, $\omega_{d-2} \in \mathtt{Pol}_{d-2}(E) \subset \mathtt{Pol}_{d-1}(E)$. This contradicts Corollary 3.1.8. □

Let us pay an attention to some dimension formulae. It follows from (3.22), (3.20) and (3.21) that

$$\dim \mathtt{Pol}_d(E) = \dim \mathcal{P}_d(E) - \dim \mathcal{P}_{d-2}(E).$$

Now by (3.25)

$$\dim \mathtt{Pol}_d(E) = \dim \mathcal{P}(E; d-1) + \dim \mathcal{P}(E; d). \tag{3.37}$$

The summands in (3.37) can be easily calculated. Indeed, $\dim \mathcal{P}(E; d)$ is the total number of multiindices $I = (i_1, \ldots, i_m)$ with $|I| = i_1 + \ldots + i_m = d$. It is well known from the elementary combinatorics that this number is the binomial coefficient $\binom{m+d-1}{m-1}$. Hence,

$$\boxed{\dim \mathcal{P}(E; d) = \binom{m+d-1}{m-1}}\ . \tag{3.38}$$

By (3.37) we obtain the dimension formula

$$\boxed{\dim \mathtt{Pol}_d(E) = \binom{m+d-2}{m-1} + \binom{m+d-1}{m-1}}\ . \tag{3.39}$$

Combining this result with the formula

$$\dim \mathtt{Harm}(E; d) = \dim \mathtt{Pol}_d(E) - \dim \mathtt{Pol}_{d-1}(E) \tag{3.40}$$

(see (3.35)), we obtain *the classical formula for the dimension of the space of spherical harmonics of degree d,*

$$\boxed{h_{m,d} \equiv \dim \mathtt{Harm}(E; d) = \binom{m+d-1}{m-1} - \binom{m+d-3}{m-1}}\ . \tag{3.41}$$

Certainly, the dimension of the space $\mathcal{H}(E;d)$ of harmonic forms of degree d on E is the same.

Formula (3.37) suggests us to consider the homomorphism

$$r|\Big(\mathcal{P}(E;d-1)\dot{+}\mathcal{P}(E;d)\Big) \tag{3.42}$$

which maps its domain into $\texttt{Pol}_d(E)$. Let us show that *(3.42) is an isomorphism.* By (3.37) and Corollary 1.7.3 it is sufficient to prove that the kernel of (3.42) is zero. Suppose that some nonzero forms $\psi_d \in \mathcal{P}(E;d)$ and $\psi_{d-1} \in \mathcal{P}(E;d-1)$ coincide on the unit sphere. Then

$$\psi_d\Big(\frac{x}{\|x\|}\Big) = \psi_{d-1}\Big(\frac{x}{\|x\|}\Big), \quad x \in E \setminus \{0\}, \tag{3.43}$$

whence

$$\psi_d(x) = \|x\|\psi_{d-1}(x), \quad x \in E \tag{3.44}$$

by homogeneity. This is a contradiction. Indeed, (3.44) implies that the polynomial ψ_d^2 contains the irreducible factor $\langle x,x\rangle$ in an odd power.

Thus, we have proved the

LEMMA 3.1.10. $\texttt{Pol}_d(E) \approx \mathcal{P}(E;d-1)\dot{+}\mathcal{P}(E;d)$ *by restriction (3.42).*

The inverse isomorphism is the direct sum of the homogeneous liftings from $\mathcal{P}(E;d-1)$ and $\mathcal{P}(E;d)$. As a rule, this lifting is not harmonic one.

EXAMPLE 3.1.11.For $\phi(x) \equiv 1$ the lifting is $\langle x,x\rangle^{\frac{d}{2}}$ if d is even but $\langle x,x\rangle^{\frac{d-1}{2}}$ if d is odd; the harmonic lifting is $\phi^{(h)}(x) \equiv 1$. The number $\deg\phi = 0$ is independent of d. $\square$

Note that Lemma 3.1.10 can be proved more directly, with no references to the theory of harmonic functions and its consequences such as (3.37), in particular. Then (3.37) follows from Lemma 3.1.10. As a result we get (3.39) and then (3.41) as before.

The alternative proof of Lemma 3.1.10 we mean requires to show directly that the homomorphism (3.42) is surjective (since its injectivity has been already proved).

Let $\phi \in \texttt{Pol}_d(E)$, $\quad \phi = r\psi$, $\quad \deg\psi \le d$. According to (3.25)

$$\psi(x) = \sum_{k=0}^{[\frac{d}{2}]} \psi_{2k}(x) + \sum_{k=0}^{[\frac{d-1}{2}]} \psi_{2k+1}(x),$$

where $[\cdot]$ means the entire part. The polynomial

$$\widetilde{\psi}(x) = \sum_{k=0}^{[\frac{d}{2}]} \langle x,x\rangle^{[\frac{d}{2}]-k}\psi_{2k}(x) + \sum_{k=0}^{[\frac{d-1}{2}]} \langle x,x\rangle^{[\frac{d-1}{2}]-k}\psi_{2k+1}(x),$$

is the sum of two forms of degrees $2\left[\frac{d}{2}\right]$ and $2\left[\frac{d-1}{2}\right]+1$ respectively. Obviously,

$$2\left[\frac{d}{2}\right] = \begin{cases} d & (d \text{ is even}) \\ d-1 & (d \text{ is odd}) \end{cases} \tag{3.45}$$

and

$$2\left[\frac{d-1}{2}\right]+1=\begin{cases} d-1 & (d \text{ is even}) \\ d & (d \text{ is odd}) \end{cases} . \tag{3.46}$$

Since $r\widetilde{\psi}=\phi$, Lemma 3.1.10 is proven once again.

In many situations the cases of even and odd d should be consider separately like before but the notation

$$\boxed{\varepsilon_d = \mathrm{res}(d)(\mathrm{mod}\ 2)} , \tag{3.47}$$

allows us to unify the exposition. In particular, this regards to the subspaces

$$\mathtt{Pol}(E;d) = r\mathcal{P}(E;d) \subset \mathtt{Pol}_d(E). \tag{3.48}$$

By the homogeneous lifting (3.6) we have

$$\mathtt{Pol}(E;d) \approx \mathcal{P}(E;d). \tag{3.49}$$

PROPOSITION 3.1.12. *For* $\mathtt{Pol}(E;d)$ *the harmonic decomposition is*

$$\boxed{\mathtt{Pol}(E;d) = \mathtt{Harm}(E;\varepsilon_d)\dot{+}\mathtt{Harm}(E;\varepsilon_d+2)\dot{+}\ldots\dot{+}\mathtt{Harm}(E;d)} . \tag{3.50}$$

Before to prove this we introduce one more notation,

$$\boxed{\mathcal{E}_d = \{\varepsilon_d, \varepsilon_d+2, \ldots, d\} = \{k:\ 0 \le k \le d,\ k \equiv d(\mathrm{mod}\ 2)\}} . \tag{3.51}$$

Proof. We need to prove that for $\phi \in \mathtt{Pol}(E;d)$ the harmonic decomposition

$$\phi(x) = \sum_{k=0}^{d} \phi_k(x) \tag{3.52}$$

does not contain nonzero summands ϕ_k with $k \notin \mathcal{E}_d$.

Let $\phi = r\psi,\quad \psi \in \mathcal{P}(E;d)$. Obviously, $\phi(-x) = (-1)^d\phi(x)$, since ψ has the same property. This also is preserved by the harmonic lifting, i.e. $\phi^{(h)}(-x) = (-1)^d\phi^{(h)}(x)$ since $\phi^{(h)}(-x)$ is harmonic together with $\phi^{(h)}(x)$. The harmonic lifting for (3.52) is

$$\phi^{(h)}(x) = \sum_{k=0}^{d} \phi_k^{(h)}(x),$$

where $\phi_k^{(h)} \in \mathcal{H}(E;k)$. Hence,

$$\phi^{(h)}(-x) = \sum_{k=0}^{d} (-1)^k \phi_k^{(h)}(x).$$

We see that $(-1)^d\phi_k^{(h)} = (-1)^k\phi_k^{(h)}$, so $\phi_k^{(h)} = 0$, since $k \notin \mathcal{E}_d$. Thus, $\phi_k = 0$ for such k.□

COROLLARY 3.1.13. *The direct decomposition*

$$\boxed{\mathtt{Pol}_d(E) = \mathtt{Pol}(E; d-1) \dot{+} \mathtt{Pol}(E; d)} \tag{3.53}$$

holds

The following commutative diagram illustrates the relations between the functional spaces we have considered in this Section:

$$
\begin{array}{ccccccc}
 & & & \approx & & & \\
\mathcal{H}(E;d) & \subset & \mathcal{P}(E;d) & \xrightarrow[\approx]{} & \mathtt{Pol}(E;d) & \supset & \mathtt{Harm}(E;d) \\
\cap & & \cap & & \cap & & \cap \\
\mathcal{H}_d(E) & \subset & \mathcal{P}_d(E) & \to & \mathtt{Pol}_d(E) & = & \mathtt{Harm}_d(E) \\
\cap & & \cap & & \cap & & \cap \\
\mathcal{H}(E) & \subset & \mathcal{P}(E) & \to & \mathtt{Pol}(E) & = & \mathtt{Harm}(E) \\
 & & & \approx & & &
\end{array}
$$

All arrows here mean the restriction homomorphisms. In the cases of isomorphisms the inverse arrows are the corresponding liftings (homogeneous or harmonic).

Now we provide the space $\mathtt{Pol}(E)$ of polynomial functions on E with the inner product

$$(\phi_1, \phi_2) = \int \overline{\phi_1}\phi_2 \, d\sigma \equiv \int_{\mathbf{S}(E)} \overline{\phi_1}\phi_2 \, d\sigma, \tag{3.54}$$

where σ is the normalized Lebesgue measure on $\mathbf{S}(E)$. In particular,

$$\boxed{(\mathbf{1}, \phi) = \textstyle\int \phi \, d\sigma} \tag{3.55}$$

where $\mathbf{1}(x) = 1$. Note that $\mathbf{1} \in \mathtt{Harm}(E; 0)$ and $h_{m,0} = 1$ according to (3.41), hence, $(\mathbf{1})$ is a basis in $\mathtt{Harm}(E; 0)$, so that $\mathtt{Harm}(E; 0)$ consists of constant functions. **`Below we do not distinguish between the constant functions and the corresponding scalars.`**

It follows from (3.55) that

$$\|\mathbf{1}\| = \sqrt{(\mathbf{1}, \mathbf{1})} = 1 \tag{3.56}$$

since the measure σ is normalized.

PROPOSITION 3.1.14. *For any* $\phi \in \mathtt{Pol}_d(E)$ *its harmonic component* ϕ_0*,* $\deg \phi_0 = 0$*, is*

$$\boxed{\phi_0 = (\mathbf{1}, \phi)} \quad . \tag{3.57}$$

Proof. Let

$$\phi = \sum_{k=0}^{d} \phi_k$$

be the harmonic decomposition of ϕ. Then its harmonic lifting is

$$\phi^{(h)} = \sum_{k=0}^{d} \phi_k^{(h)}.$$

We have

$$\int \phi\, d\sigma = \int \phi^{(h)}\, d\sigma = \sum_{k=0}^{d} \int \phi_k^{(h)}\, d\sigma = \int \phi_0^{(h)}\, d\sigma = \int \phi_0\, d\sigma \tag{3.58}$$

since

$$\int \phi_k^{(h)}\, d\sigma = \phi_k(0) = 0, \quad 1 \le k \le d, \tag{3.59}$$

according to the well-known property of harmonic functions. One can rewrite (3.58) as (3.57) because of (3.55) and $\phi_0 = \text{const}$. □

REMARK 3.1.15. Formula (3.57) can be also written as

$$\int \phi\, d\sigma = \phi^{(h)}(0). \tag{3.60}$$

Indeed, the harmonic lifting of a constant is the same constant. □

The space $\texttt{Harm}(E;k)$ is invariant with respect to the natural action of the orthogonal group $O(E)$:

$$(g\phi)(x) = \phi(gx), \quad g \in O(E), \tag{3.61}$$

because of (3.12). Actually, (3.61) defines a representation of the group $O(E)$ in $\texttt{Harm}(E;k)$. This representation is unitary since the measure σ is orthogonally invariant. (This is the Haar measure on the unit sphere $\mathbf{S}(E)$.)

Let $(S_{ki})_{i=1}^{h_{m,k}}$ be an orthonormal basis in $\texttt{Harm}(E;k)$. (Recall that $h_{m,k} = \dim \texttt{Harm}(E;k)$ in notation (3.41).) The classical Addition Theorem shows that the function

$$f(x,y) = \sum_{i=1}^{h_{m,k}} \overline{S_{ki}(x)} S_{ki}(y) \tag{3.62}$$

of $x, y \in \mathbf{S}(E)$ is independent of the choice of the basis, and, moreover, it is a special polynomial of $\langle x, y\rangle$. The first of these properties can be easily checked. Indeed, if $(T_{ki})_{i=1}^{h_{m,k}}$ is one more orthonormal basis in $\texttt{Harm}(E;k)$, then

$$T_{ki}(x) = \sum_{j=1}^{h_{m,k}} \alpha_{ji} S_{kj}(x), \quad 1 \le i \le h_{m,k},$$

where $[\alpha_{ji}]$ is a complex unitary matrix. Hence,

$$\sum_{i=1}^{h_{m,k}} \overline{T_{ki}(x)} T_{ki}(y) = \sum_{j,l=1}^{h_{m,k}} \overline{S_{kj}(x)} \left(\sum_{i=1}^{h_{m,k}} \overline{\alpha}_{ji} \alpha_{li} \right) S_{kl}(y) = \sum_{j,l=1}^{h_{m,k}} \overline{S_{kj}(x)} \delta_{jl} S_{kl}(y) = \sum_{j=1}^{h_{m,k}} \overline{S_{kj}(x)} S_{kj}(y).$$

In particular, *the function $f(x,y)$ is unitary invariant*, i.e.

$$f(gx, gy) = f(x, y) \tag{3.63}$$

for all $g \in O(E)$. Indeed, for any $g \in O(E)$ the functions gS_{ki}, $1 \leq i \leq h_{m,k}$, form an orthonormal basis in $\mathtt{Harm}(E;k)$ because of invariance of the measure σ.

By Theorem 1.9.32

$$f(x, y) = Q_{m,k}(\langle x, y\rangle) \tag{3.64}$$

where $Q_{m,k}$ is a function $[-1,1] \to \mathbf{C}$. (The range of $\langle x,y\rangle$ on $\mathbf{S}(E) \times \mathbf{S}(E)$ contains in $[-1,1]$ by Schwartz's inequality and all values from $[-1,1]$ are attained already on the real vectors.) Thus,

$$\boxed{\sum_{i=1}^{h_{m,k}} \overline{S_{ki}(x)} S_{ki}(y) = Q_{m,k}(\langle x, y\rangle)} \tag{3.65}$$

by (3.64) and (3.62). (Recall that any function $\phi : \mathbf{S}(E) \to \mathbf{C}$ which only depends on $\langle x, y\rangle$ with fixed $y \in \mathbf{S}(E)$ is called **zonal** with respect to the **pole** y.)

LEMMA 3.1.16. *$Q_{m,k}$ is a polynomial of degree $\leq k$.*

Proof. Further y_0 is a fixed point (the "North Pole") on the unit sphere $\mathbf{S}(E)$. Choose $x_0 \in \mathbf{S}(E)$ such that $x_0 \perp y_0$ and consider the circle

$$x(\vartheta) = y_0 \cos\vartheta + x_0 \sin\vartheta, \quad 0 \leq \vartheta < 2\pi,$$

on the sphere. Obviously,

$$\langle x(\vartheta), y_0\rangle = \cos\vartheta. \tag{3.66}$$

By the lifting $\mathtt{Harm}(E;k) \to \mathcal{H}(E;k)$ we get

$$S_{ki}(x(\vartheta)) = S_{ki}^{(h)}(y_0 + x_0 \tan\vartheta)\cos^k\vartheta = \sum_{j=0}^{k} b_{ki,j} \tan^j\vartheta \cdot \cos^k\vartheta = \sum_{j=0}^{k} b_{ki,j} \sin^j\vartheta \cdot \cos^{k-j}\vartheta, \tag{3.67}$$

where $b_{ki,j}$ are some complex coefficients. According to (3.65), (3.66) and (3.67)

$$Q_{m,k}(\cos\vartheta) = \sum_{i=1}^{h_{m,k}} \overline{S_{ki}(x(\vartheta))} S_{ki}(y_0) = \sum_{j=0}^{k} a_{kj} \sin^j\vartheta \cdot \cos^{k-j}\vartheta$$

with some new coefficients a_{kj}. By substitution $\vartheta \mapsto 2\pi - \vartheta$

$$Q_{m,k}(\cos\vartheta) = \sum_{j=0}^{k} (-1)^j a_{kj} \sin^j\vartheta \cdot \cos^{k-j}\vartheta.$$

Hence, $(-1)^j a_{kj} = a_{kj}$, i.e. $a_{kj} = 0$ for odd j. Thus,

$$Q_{m,k}(u) = \sum_{j=0}^{[\frac{k}{2}]} a_{k,2j}(1-u^2)^j u^{k-2j}, \quad -1 \leq u \leq 1. \qquad \square$$

The polynomial $Q_{m,k}(u)$ can be expressed in terms of the Jacobi polynomials. Namely, the following **Addition Theorem** is valid.

THEOREM 3.1.17. *For any orthonormal basis* $(S_{kj})_{j=1}^{h_{m,k}}$ *in the space* $\mathtt{Harm}(E;k)$ *of spherical harmonics of degree* k *the* **Addition Formula**

$$\boxed{\sum_{j=1}^{h_{m,k}} \overline{S_{kj}(x)} S_{kj}(y) = a_{m,k} C_k^{\frac{m-2}{2}}(\langle x,y\rangle)} \quad , \tag{3.68}$$

holds with the coefficient

$$\boxed{a_{m,k} = \frac{h_{m,k}}{C_k^{\frac{m-2}{2}}(1)} = \frac{\binom{m+k-1}{m-1} - \binom{m+k-3}{m-1}}{\binom{m+k-3}{m-3}}} \quad . \tag{3.69}$$

Proof. We already have (3.68) in the preliminary form (3.65). We see that for any fixed $y \in \mathbf{S}(E)$ the function $Q_{m,k}(\langle x,y\rangle)$ belongs to $\mathtt{Harm}(E;k)$. Its homogeneous lifting is $\|x\|^k Q_{m,k}(w_y(x))$ where $w_y(x) = \langle x,y\rangle/\|x\|, \quad x \in E$. Indeed,

$$\|x\|^k Q_{m,k}(w_y(x)) = \|x\|^k Q_{m,k}\Big(\frac{\langle x,y\rangle}{\|x\|}\Big) = \|x\|^k \sum_{j=1}^{h_{m,k}} \overline{S_{kj}\Big(\frac{x}{\|x\|}\Big)} S_{kj}(y) = \sum_{j=1}^{h_{m,k}} \overline{S_{kj}^{(h)}(x)} S_{kj}(y).$$

The latter function of x belongs to $\mathcal{H}(E;k)$, i.e. it is a harmonic form of degree k. By Lemma 2.3.2

$$(1-u^2)Q''_{m,k}(u) + (1-m)uQ'_{m,k}(u) + k(k+m-2)Q_{m,k}(u) = 0, \quad -1 \le u \le 1.$$

All polynomial solutions of this **Jacobi's differential equation** are proportional to $C_k^{\frac{m-2}{2}}(u)$ (see [54, Theorem 4.2.1]). It remains to determine the coefficient $a_{m,k}$ in (3.68), i.e. in the relation

$$\boxed{Q_{m,k}(u) = a_{m,k} C_k^{\frac{m-2}{2}}(u)} \quad . \tag{3.70}$$

Setting $y = x$ in (3.68) we get

$$\sum_{j=1}^{h_{m,k}} |S_{kj}(x)|^2 = Q_{m,k}(1). \tag{3.71}$$

Hence,

$$\boxed{h_{m,k} = Q_{m,k}(1)} \tag{3.72}$$

by integration of (3.71) over the sphere $\mathbf{S}(E)$. This yields the first equality in (3.69) which, in turn, implies the second one by (3.41) and (2.7). □

By the way, (3.70) shows that

$$\boxed{\deg Q_{m,k} = \deg C_k^{\frac{m-2}{2}} = k} \tag{3.73}$$

since $a_{m,k} \neq 0$ by (3.69). In particular,

$$\boxed{Q_{m,0}(u) = 1} \tag{3.74}$$

since $Q_{m,0}$ is a constant and $Q_{m,0}(1) = h_{m,0} = 1$. In addition, as follows from (3.70) *the polynomial* $Q_{m,k}(u)$ *is even for even* k *and odd for odd* k.

The Addition Theorem is the most fundamental fact from the theory of spherical harmonics. Its numerous consequences can be encountered in many areas of Mathematics. The first consequence of the Addition Theorem is the following important fact of the Unitary Representations Theory where this follows from the irreducibility of the representation (3.61) in $\mathtt{Harm}(E;k)$.

THEOREM 3.1.18. *The harmonic decomposition (3.31) is orthogonal, i.e.*

$$\boxed{\mathtt{Harm}(E;k) \perp \mathtt{Harm}(E;l) \quad (k \neq l)} \quad . \tag{3.75}$$

We prove Theorem 3.1.18 using orthogonality of the Gegenbauer polynomials and the integration formula (2.26). First of all, (2.26) immediately yields

LEMMA 3.1.19 *The inner product of two zonal functions* $\theta_1(\langle x,y\rangle)$ *and* $\theta_2(\langle x,y\rangle)$ *of the sphere is reduced according to*

$$\boxed{\int \overline{\theta_1(\langle x,y\rangle)}\theta_2(\langle x,y\rangle)\,d\sigma(y) = \int_{-1}^{1} \overline{\theta_1(u)}\theta_2(u)\Omega_m(u)\,du} \tag{3.76}$$

where the normalized weight Ω_m *is defined according to (2.10).*

On the right hand side of (3.76) we have the inner product,

$$(\theta_1,\theta_2)_{\Omega_m} = \int_{-1}^{1} \overline{\theta_1(u)}\theta_2(u)\Omega_m(u)\,du, \tag{3.77}$$

and the lemma states that

$$\boxed{(\theta_1(\langle x,\cdot\rangle),\theta_2(\langle x,\cdot\rangle)) = (\theta_1,\theta_2)_{\Omega_m}} \quad . \tag{3.78}$$

In order to prove Theorem 3.1.18 we apply (3.78) to $\theta_1 = Q_{m,k}$ and $\theta_2 = Q_{m,l}$ and get

$$\boxed{(Q_{m,k}(\langle x,\cdot\rangle), Q_{m,l}(\langle x,\cdot\rangle)) = (Q_{m,k}, Q_{m,l})_{\Omega_m}} \quad . \tag{3.79}$$

This yields

$$\int Q_{m,k}(\langle x,y\rangle) Q_{m,l}(\langle x,y\rangle)\, d\sigma(y) = 0, \quad k \neq l, \tag{3.80}$$

by (3.70) and orthogonality of the Gegenbauer polynomials. By the next integration

$$\int\int Q_{m,k}(\langle x,y\rangle) Q_{m,l}(\langle x,y\rangle)\, d\sigma(x)\, d\sigma(y) = 0, \quad k \neq l. \tag{3.81}$$

On the other hand, the Addition Theorem yields

$$\begin{aligned} &\int\int Q_{m,k}(\langle x,y\rangle) Q_{m,l}(\langle x,y\rangle)\, d\sigma(x)\, d\sigma(y) = \\ &\sum_{j=1}^{h_{m,k}} \sum_{i=1}^{h_{m,l}} \int S_{kj}(x)\overline{S_{li}(x)}\, d\sigma(x) \int \overline{S_{kj}(y)} S_{li}(y)\, d\sigma(y) = \\ &\sum_{j=1}^{h_{m,k}} \sum_{i=1}^{h_{m,l}} \overline{(S_{kj}, S_{li})}(S_{kj}, S_{li}) = \sum_{j=1}^{h_{m,k}} \sum_{i=1}^{h_{m,l}} |(S_{kj}, S_{li})|^2. \end{aligned} \tag{3.82}$$

After substitution in (3.81) we conclude that

$$(S_{kj}, S_{li}) = 0 \quad (k \neq l, 1 \leq j \leq h_{m,k}, \quad 1 \leq i \leq h_{l,k})$$

which imply (3.75) since $(S_{kj})_1^{h_{m,k}}$ and $(S_{lj})_1^{h_{m,k}}$ are the bases in $\mathtt{Harm}(E;k)$ and $\mathtt{Harm}(E;l)$ respectively. Theorem (3.1.18) is proved.

Now the harmonic decomposition (3.31) can be specified as

$$\boxed{\mathtt{Pol}_d(E) = \mathtt{Harm}(E;0) \bigoplus \mathtt{Harm}(E;1) \bigoplus \ldots \bigoplus \mathtt{Harm}(E;d)} \tag{3.83}$$

In particular, this shows that the orthoprojection $\phi \mapsto (\mathbf{1}, \phi)$ of the space $\mathtt{Pol}_d(E)$ onto $\mathtt{Harm}(E;0)$ (see Proposition 3.1.14) annihilates all $\mathtt{Harm}(E;k)$ with $k \geq 1$.

As a consequence of (3.83) and (3.50)

$$\boxed{\mathtt{Pol}(E;d) = \mathtt{Harm}(E;\varepsilon_d) \bigoplus \mathtt{Harm}(E;\varepsilon_d + 2) \bigoplus \ldots \bigoplus \mathtt{Harm}(E;d)} \tag{3.84}$$

and then Corollary 3.1.13 can be specified as

$$\boxed{\mathtt{Pol}_d(E) = \mathtt{Pol}(E;d-1) \bigoplus \mathtt{Pol}(E;d) \quad .} \tag{3.85}$$

Now we consider the linear space Π of all polynomials of one variable. In the Euclidean space (Π, Ω_m) the polynomials $Q_{m,k}$ form an orthogonal basis which is compatible with the filtration

$$\Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \ldots, \quad \bigcup_{d=0}^{\infty} \Pi_d = \Pi.$$

where $\Pi_d = \{Q \in \Pi : \deg Q \leq d\}$ (cf. Example 1.9.15).

Let $\Pi^{(0)}$ and $\Pi^{(1)}$ be the subspaces of even and odd polynomials respectively. Then

$$\boxed{\Pi = \Pi^{(0)} \bigoplus \Pi^{(1)}} \tag{3.86}$$

since the weight Ω_m is even. Respectively,

$$\Pi_d = \Pi_d^{(0)} \oplus \Pi_d^{(1)} \tag{3.87}$$

where

$$\boxed{\Pi_d^{(\varepsilon)} = \Pi \cap \Pi_d, \ \varepsilon \in \{0, 1\}} \ . \tag{3.88}$$

For any d the system

$$(Q_{m,k} : \ 0 \leq k \leq d, \ k \equiv \varepsilon (\text{mod } 2))$$

is an orthogonal basis in $(\Pi_d^{(\varepsilon)}, \Omega_m)$.

The decomposition of a polynomial $F \in \Pi_d$ for the basis $(Q_{m,k})_0^d$ is

$$F = \sum_{k=0}^{d} c_{m,k}(F) Q_{m,k} \ , \tag{3.89}$$

where

$$c_{m,k}(F) = \frac{(Q_{m,k} \ , F)_{\Omega_m}}{(Q_{m,k} \ , Q_{m,k})_{\Omega_m}} = \frac{(Q_{m,k} \ , F)_{\Omega_m}}{\|Q_{m,k}\|^2_{\Omega_m}}, \quad 0 \leq k \leq d, \tag{3.90}$$

are the Fourier coefficients (cf. (1.71)). Formally,

$$F = \sum_{k=0}^{\infty} c_{m,k}(F) Q_{m,k}$$

where $c_{m,k}(F) = 0$ for $k > \deg F$. Then

$$\deg F = \max\{k : c_{m,k}(F) \neq 0\}. \tag{3.91}$$

PROPOSITION 3.1.20. *The formula*

$$\boxed{\|Q_{m,k}\|^2_{\Omega_m} = Q_{m,k}(1)} \tag{3.92}$$

holds.

Proof. It follows from (3.79) with $k = l$

$$\|Q_{m,k}\|^2_{\Omega_m} = \int Q^2_{m,k}(\langle x, y\rangle)\, d\sigma(y). \tag{3.93}$$

The latter integral can be calculated by the Addition Formula which is, in fact, the Fourier decomposition of $Q_{m,k}(\langle x, \cdot\rangle)$ for the orthonormal basis (S_{ki}) in $\mathtt{Harm}(E; k)$. The corresponding Fourier coefficients are $S_{ki}(x)$, so that

$$\int Q^2_{m,k}(\langle x, y\rangle)\, d\sigma(y) = \sum_{i=1}^{h_{m,k}} |S_{ki}(x)|^2 = Q_{m,k}(1) \tag{3.94}$$

by (3.71). It remains to compare (3.94) to (3.93). □

Proposition 3.1.20 combining with (3.72) implies

COROLLARY 3.1.21. *The formula*

$$\boxed{\|Q_{m,k}\|^2_{\Omega_m} = h_{m,k}} \tag{3.95}$$

holds.

In particular,

$$\boxed{\|Q_{m,0}\|^2_{\Omega_m} = 1} \tag{3.96}$$

hence,

$$\boxed{c_{m,0}(F) = \int_{-1}^{1} F(u)\Omega_m(u)\, du} \tag{3.97}$$

according to (3.90).

The Addition Theorem is very helpful to study some metric properties of finite subsets $X \subset \mathbf{S}(E)$ we are interested in. At this point we derive an identity which relates to the Hermitian quadratic form

$$\sum_{x,y\in X} F(\langle x, y\rangle)\overline{\lambda(x)}\lambda(y) \tag{3.98}$$

where F is a polynomial and $\lambda : X \to \mathbf{C}$ is a function.

LEMMA 3.1.22. *The equality*

$$\boxed{\sum_{x,y\in X} F(\langle x, y\rangle)\overline{\lambda(x)}\lambda(y) = \sum_{k=0}^{\deg F} c_{m,k}(F) \sum_{i=1}^{h_{m,k}} \left| \sum_{x\in X} S_{ki}(x)\lambda(x)\right|^2} \tag{3.99}$$

holds.

Actually, this is a decomposition of the form (3.98) into a sum of squares of linear forms.

Proof. By (3.89)

$$\sum_{x,y\in X} F(\langle x,y\rangle)\overline{\lambda(x)}\lambda(y) = \sum_{k=0}^{\deg F} c_{m,k}(F) \sum_{x,y\in X} Q_{m,k}(\langle x,y\rangle)\overline{\lambda(x)}\lambda(y) \tag{3.100}$$

and by the Addition Theorem

$$\sum_{x,y\in X} Q_{m,k}(\langle x,y\rangle)\overline{\lambda(x)}\lambda(y) = \sum_{i=1}^{h_{m,k}} \sum_{x,y\in X} \overline{S_{ki}(x)}S_{ki}(y)\overline{\lambda(x)}\lambda(y) = \sum_{i=1}^{h_{m,k}} \left|\sum_{x\in X} S_{ki}(x)\lambda(x)\right|^2. \tag{3.101}$$

□

3.2 Polynomial functions on the projective spaces

Recall that the projective space $\mathbf{P}(E)$ over a field $\mathbf{K}$ is the set of one-dimensional subspaces of the space E. In our context E is a m-dimensional right linear Euclidean space over $\mathbf{K} = \mathbf{R}$, or $\mathbf{C}$, or $\mathbf{H}$. The number $m-1$ is called the **dimension** of $\mathbf{P}(E)$. In particular, if $E = \ell^m_{2,\mathbf{K}}$ then $\mathbf{P}(E)$ is denoted by $\mathbf{KP}^{m-1}$ and called the **arithmetic projective space**. A point $\mathbf{x} \in \mathbf{P}(E)$ is defined by any nonzero vector $x \in \mathbf{x}$, $x \neq 0$. The natural mapping $x \mapsto \mathbf{x}$ from $E \setminus \{0\}$ onto $\mathbf{P}(E)$ is called the **projectivization**.

If $(e_k)_1^m$ is a basis in E, the coordinates of a vector $x \in E$ are called the **homogeneous coordinates** of the corresponding point $\mathbf{x} \in \mathbf{P}(E)$. The homogeneous coordinates are defined up to proportionality. So, if the coordinate column of x is

$$[\xi] = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}$$

then one can denote the m-tuple of homogeneous coordinates of $\mathbf{x}$ by $[\xi_1 : \ldots : \xi_m]$. Formally, the latter is the class of the equivalence such that $[\xi] \equiv [\eta]$ if and only if the nonzero columns $[\xi]$, $[\eta]$ are proportional.

The following description of the projective space $\mathbf{P}(E)$ is preferable for our purposes.

Let us denote by $U(\mathbf{K})$ the multiplicative group of scalars of modulus 1, so that $U(\mathbf{R}) = \mathbf{Z}_2 = \{-e, e\}$, $U(\mathbf{C}) = \mathbf{S}^1$ (the unit sphere in $\mathbf{C}$, in fact, the unit circle), $U(\mathbf{H}) = \mathbf{S}^3$ (the unit sphere in $\mathbf{H}$). By the way, $U(\mathbf{K})$ is just the group $U_1(\mathbf{K})$ of 1×1 unitary matrices over $\mathbf{K}$. Also note that the group $U(\mathbf{H})$ is isomorphic to the group $SU(2)$ of the complex unitary matrices with determinant1. The isomorphism is

$$\xi + \eta\mathbf{j} \mapsto \begin{pmatrix} \xi & -\eta \\ \overline{\eta} & \overline{\xi} \end{pmatrix}, \quad |\xi|^2 + |\eta|^2 = 1.$$

The group $U(\mathbf{K})$ acts on the unit sphere $\mathbf{S}(E)$ (and even in the whole space) by multiplication: $x \mapsto x\gamma$, $\gamma \in U(\mathbf{K})$. In E this action is $\mathbf{R}$-linear (but not $\mathbf{K}$-linear if $\mathbf{K}$ is not commutative).

Consider the quotient space $\mathbf{S}(E)/U(\mathbf{K})$ of this action. Its elements are the orbits of vectors $x \in \mathbf{S}(E)$,

$$\mathrm{Orb}(x) = U(\mathbf{K})x = \{y : y = x\gamma, \gamma \in U(\mathbf{K})\}.$$

All elements of an orbit define the same point in $\mathbf{P}(E)$. Thus, we have the natural mapping $\mathbf{S}(E)/U(\mathbf{K}) \to \mathbf{P}(E)$. This mapping is bijective, the inverse mapping is correctly defined as $\mathbf{x} \mapsto \mathrm{Orb}\,(\widehat{x})$ with any $x \in \mathbf{x},\ x \neq 0$ (Recall that $\widehat{x} = x/\|x\|$). In such a way the projective space $\mathbf{P}(E)$ can be identified with the quotient space $\mathbf{S}(E)/U(\mathbf{K})$,

$$\boxed{\mathbf{P}(E) \equiv \mathbf{S}(E)/U(\mathbf{K})}\ . \tag{3.102}$$

`Below we use the latter as a natural domain for definition of polynomial functions on the projective space.`

A function $\phi : \mathbf{S}(E) \to \mathbf{C}$ is called a **polynomial function over K** if

1) ϕ is a polynomial function on the real unit sphere $\mathbf{S}(E_{\mathbf{R}}) = \mathbf{S}(E)$ where $E_{\mathbf{R}}$ is the realification of the space E;

2) $\phi(x\gamma) = \phi(x)$ for all $\gamma \in U(\mathbf{K})$. In particular, $\phi(-x) = \phi(x)$, i.e. ϕ is an even function.

Recall that **1)** means that $\phi = \psi|\mathbf{S}(E)$ where ψ is a polynomial on $E_{\mathbf{R}}$.

Condition **2)** means that the value $\phi(x)$ only depends on the corresponding point $\mathbf{x} \in \mathbf{P}(E)$. Thus, *all polynomial functions over* **K** *can be considered as functions on the projective space.* However, for our purposes we will keep them on $\mathbf{S}(E)$. In particular, $|\langle x, y\rangle|^2$ with $x, y \in \mathbf{S}(E)$ is well-defined on the projective space and this is a polynomial function of x and of y separately.

The set $\mathtt{Pol}_{\mathbf{K}}(E)$ of polynomial functions on the projective space $\mathbf{P}(E)$ is a subset of $\mathtt{Pol}(E_{\mathbf{R}})$. In fact, this is a subspace (a subalgebra) of the linear space (algebra) $\mathtt{Pol}(E_{\mathbf{R}})$.

Now we consider the linear space $\mathcal{P}(E_{\mathbf{R}})$ of polynomials on $E_{\mathbf{R}}$ and its subspace $\mathcal{P}_{\mathbf{K}}(E)$ of the $U(\mathbf{K})$-invariant polynomials:

$$\mathcal{P}_{\mathbf{K}(E)} = \{\psi \in \mathcal{P}(E) : \quad \psi(x\gamma) = \psi(x),\ x \in E, \gamma \in U(\mathbf{K})\}. \tag{3.103}$$

If $\mathbf{K} = \mathbf{R}$ then $E = E_{\mathbf{R}}$, $\mathcal{P}_{\mathbf{K}}(E)$ consists of all even polynomials,

$$\mathcal{P}_{\mathbf{R}}(E) = \{\psi \in \mathcal{P}(E) : \psi(-x) = \psi(x)\}. \tag{3.104}$$

For any $\mathbf{K}$ one can say that all functions from $\mathcal{P}_{\mathbf{K}}(E)$ are even but the converse is not true in general.

If $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$ then some functions from $\mathcal{P}_{\mathbf{K}}(E)$ are not polynomials on E. For example, this relates to $\langle x, x\rangle$ and $|\langle x, y\rangle|^d$ with any fixed y and any even integer $d \geq 2$. For this reason, `speaking about polynomials (in particular, about forms) we mean the polynomials (forms) on` $E_{\mathbf{R}}$.

DEFINITION 3.2.1. *Let d be an even integer, $d \geq 2$. The polynomial functions*

$$x \mapsto |\langle x, y\rangle|^d, \quad x \in \mathbf{S}(E); \quad y \in E, \tag{3.105}$$

are called the **elementary polynomial functions of degree** *d.*

Let $\mathcal{P}_{\mathbf{K}}(E;l) \subset \mathcal{P}(E_{\mathbf{R}};l)$ be the subspace of all $U(\mathbf{K})$-invariant forms of degree $l \geq 0$.

LEMMA 3.2.2.

a) *If l is odd, $l = 2k+1$, then $\mathcal{P}_{\mathbf{K}}(E;l) = 0$.*

b) *If l is even, $l = 2k$, then $\mathcal{P}_{\mathbf{K}}(E;l)$ consists of the polynomials ψ of degree l such that*

$$\psi(x\lambda) = |\lambda|^l \psi(x), \quad \lambda \in \mathbf{K}, \tag{3.106}$$

or, equivalently,

$$\psi|\mathbf{S}(E) \in \mathtt{Pol}_{\mathbf{K}}(E). \tag{3.107}$$

Proof. If $l = 2k+1$ then any $\psi \in \mathcal{P}(E_{\mathbf{R}};l)$ is odd, i.e. $\psi(-x) = -\psi(x)$ and if, in addition, ψ is $U(\mathbf{K})$-invariant then ψ is even, i.e. $\psi(-x) = \psi(x)$. Hence, $\psi = 0$.

Let $l = 2k$. Then for $\psi \in \mathcal{P}_{\mathbf{K}}(E;l)$ we get (3.106) by substitution $\lambda = \gamma|\lambda|$, $\gamma \in U(\mathbf{K})$, into $\psi(x\lambda)$. Conversely, (3.106) implies $\psi \in \mathcal{P}_{\mathbf{K}}(E;l)$ as well as (3.106)⇒(3.107). Now assuming (3.107), we have

$$\psi(x\gamma) = \|x\|^l \psi(\widehat{x}\gamma) = \|x\|^l \psi(\widehat{x}) = \psi(x)$$

for $x \neq 0$. The case $x = 0$ is trivial. □

Now we can adjust the decomposition (3.25) to the projective situation. Denote by $\mathcal{P}_{\mathbf{K};d}(E)$ the subspace of all $U(\mathbf{K})$-invariant polynomials of degrees $\leq d$, $\mathcal{P}_{\mathbf{K};d}(E) \subset \mathcal{P}_{\mathbf{K}}(E)$.

LEMMA 3.2.3. *In the space $\mathcal{P}_{\mathbf{K};d}(E)$ the direct decomposition*

$$\boxed{\mathcal{P}_{\mathbf{K};d}(E) = \mathcal{P}_{\mathbf{K}}(E;0) \dot{+} \mathcal{P}_{\mathbf{K}}(E;2) \dot{+} \ldots \dot{+} \mathcal{P}_{\mathbf{K}}(E;2\left[\tfrac{d}{2}\right])} \tag{3.108}$$

holds.

In other words, we claim that *the homogeneous components of any $U(\mathbf{K})$-invariant polynomials are $U(\mathbf{K})$-invariant.*

Proof. Let $\psi \in \mathcal{P}_{\mathbf{K}}(E;d)$ and let

$$\psi(x) = \sum_{k=0}^{\left[\frac{d}{2}\right]} \psi_{2k}(x) \tag{3.109}$$

where ψ_{2k} are homogeneous components of ψ. Then

$$\psi(x\gamma) = \sum_{k=0}^{\left[\frac{d}{2}\right]} \psi_{2k}(x\gamma) \tag{3.110}$$

since $x \mapsto x\gamma$ is a $\mathbf{R}$-linear operator. Comparing (3.110) to (3.109) we get $\psi_{2k}(x\gamma) = \psi_{2k}(x)$ for all k, $0 \leq k \leq \left[\frac{d}{2}\right]$. □

COROLLARY 3.2.4. *The degree of every $U(\mathbf{K})$-invariant polynomial is even.*

In particular, we have

COROLLARY 3.2.5. *If* $\psi \in \mathcal{P}_{\mathbf{K}}(E)$ *and* $\deg \psi \leq 1$ *then* $\psi = \mathrm{const}$.

COROLLARY 3.2.6. *The degree of every polynomial function is even.*

`Up to the end of this Section the number` d `is supposed to be even.` Let $\mathtt{Pol}_{\mathbf{K};d}(E)$ be the space of all polynomial functions of degrees $\leq d$. Lemma 3.1.10 implies

LEMMA 3.2.7. $\mathtt{Pol}_{\mathbf{K};d}(E) \approx \mathcal{P}_{\mathbf{K}}(E;d)$ *by restriction* $r : \mathcal{P}_{\mathbf{K}}(E;d) \to \mathtt{Pol}_{\mathbf{K};d}(E)$.

In other words, by $U(\mathbf{K})$-invariance, *any polynomial function of degree* $\leq d$ *is the restriction of a unique* $U(\mathbf{K})$*-invariant form of degree* d. Thus, the homogeneous lifting from the whole space $\mathtt{Pol}_{\mathbf{K};d}(E)$ to $\mathcal{P}_{\mathbf{K}}(E;d)$ is well-defined in the projective situation in contrast to the spherical one.

In the notation

$$\boxed{\mathtt{Pol}_{\mathbf{K}}(E;d) \equiv r\mathcal{P}_{\mathbf{K}}(E;d) \approx \mathcal{P}_{\mathbf{K}}(E;d)} \tag{3.111}$$

we have

COROLLARY 3.2.8. *The equality*

$$\boxed{\mathtt{Pol}_{\mathbf{K};d}(E) = \mathtt{Pol}_{\mathbf{K}}(E;d)} \tag{3.112}$$

holds.

Now we consider the harmonic lifting.

LEMMA 3.2.9. *For every* $\phi \in \mathtt{Pol}_{\mathbf{K}}(E)$ *its harmonic lifting* $\phi^{(h)} \in \mathcal{H}_{\mathbf{K}}(E)$ *is also* $U(\mathbf{K})$*-invariant,*

$$\phi^{(h)}(x\gamma) = \phi^{(h)}(x), \ \gamma \in U(\mathbf{K}). \tag{3.113}$$

Proof. The function $\psi(x) = \phi^{(h)}(x\gamma)$ is also harmonic since the $\mathbf{R}$-linear operator $x \mapsto x\gamma$, $\gamma \in U(\mathbf{K})$ belongs to the orthogonal group $O(E_{\mathbf{R}})$. After restriction to $x \in \mathbf{S}(E)$ we have

$$(\psi|\mathbf{S}(E))(x) = \phi(x\gamma) = \phi(x) = (\phi^{(h)}|\mathbf{S}(E))(x).$$

Hence, $\psi = \phi^{(h)}$ by the uniqueness of the harmonic lifting. $\square$

In this context the subspace $\mathcal{H}_{\mathbf{K}}(E) \subset \mathcal{P}_{\mathbf{K}}(E)$ of $U(\mathbf{K})$-invariant harmonic polynomials appears. Its natural subspace is

$$\mathcal{H}_{\mathbf{K};d}(E) = \mathcal{P}_{\mathbf{K};d}(E) \cap \mathcal{H}_{\mathbf{K}}(E)$$

with any $d \in \mathbf{N}$ (cf. 3.15) as well as $\mathcal{H}_{\mathbf{K}}(E;2k)$, the subspace of $U(\mathbf{K})$-invariant forms of degree $2k$, $0 \leq k \leq \frac{d}{2}$. The corresponding (by restriction) subspaces of $\mathtt{Pol}_{\mathbf{K}}(E)$ are $\mathtt{Harm}_{\mathbf{K}}(E)$, $\mathtt{Harm}_{\mathbf{K};d}(E)$

and $\mathtt{Harm}_{\mathbf{K}}(E;k)$ respectively. The latter is the space of $U(\mathbf{K})$**-invariant spherical harmonics of degree** k**.** The restriction isomorphism yields

$$\boxed{\mathtt{Harm}_{\mathbf{K}}(E;2k) \approx \mathcal{H}_{\mathbf{K}}(E;2k)} \tag{3.114}$$

(cf. (3.30)). Similarly,

$$\boxed{\mathtt{Harm}_{\mathbf{K}}(E) \approx \mathcal{H}_{\mathbf{K}}(E), \quad \mathtt{Harm}_{\mathbf{K};d}(E) \approx \mathcal{H}_{\mathbf{K};d}(E)} \tag{3.115}$$

(cf. (3.20)).

The projective counterpart of Theorem 3.1.2 is

THEOREM 3.2.10. $\mathtt{Pol}_{\mathbf{K};d}(E) = \mathtt{Harm}_{\mathbf{K};d}(E)$.

Proof. Since $\mathtt{Harm}_{\mathbf{K};d}(E) \subset \mathtt{Pol}_{\mathbf{K};d}(E)$, the required equality follows from Lemma 3.2.9. $\square$

COROLLARY 3.2.11. $\mathtt{Pol}_{\mathbf{K}}(E) = \mathtt{Harm}_{\mathbf{K}}(E)$.

The following commutative diagram presents the relations between the functional spaces over $\mathbf{K}$:

$$\mathcal{H}_{\mathbf{K}}(E;d) \xrightarrow{\ \approx\ } \mathtt{Harm}_{\mathbf{K}}(E;d)$$

$$\begin{array}{ccccccc}
\mathcal{H}_{\mathbf{K}}(E;d) & \subset & \mathcal{P}_{\mathbf{K}}(E;d) & \xrightarrow[\approx]{} & \mathtt{Pol}_{\mathbf{K}}(E;d) & \supset & \mathtt{Harm}_{\mathbf{K}}(E;d) \\
\cap & & \cap & & \| & & \cap \\
\mathcal{H}_{\mathbf{K};d}(E) & \subset & \mathcal{P}_{\mathbf{K};d}(E) & \rightarrow & \mathtt{Pol}_{\mathbf{K};d}(E) & = & \mathtt{Harm}_{\mathbf{K};d}(E) \\
\cap & & \cap & & \cap & & \cap \\
\mathcal{H}_{\mathbf{K}}(E) & \subset & \mathcal{P}_{\mathbf{K}}(E) & \rightarrow & \mathtt{Pol}_{\mathbf{K}}(E) & = & \mathtt{Harm}_{\mathbf{K}}(E)
\end{array}$$

$$\mathcal{H}_{\mathbf{K}}(E) \xrightarrow{\ \approx\ } \mathtt{Harm}_{\mathbf{K}}(E)$$

All these spaces are the $U(\mathbf{K})$*-invariant parts of the corresponding spaces on the sphere* $S(E_{\mathbf{R}})$*. If* $\mathbf{K} = \mathbf{R}$ *then the regarding functional spaces are the same as for the spherical case with even functions under consideration only.*

By Lemmas 3.1.6 and 3.2.9 the harmonic specialization of (3.108) is

$$\boxed{\mathcal{H}_{\mathbf{K};d}(E) = \mathcal{H}_{\mathbf{K}}(E;0)\dot{+}\mathcal{H}_{\mathbf{K}}(E;2)\dot{+}\ldots\dot{+}\mathcal{H}_{\mathbf{K}}(E;d)}\ . \tag{3.116}$$

Finally, we can obtain the following $U(\mathbf{K})$-invariant version of (3.83).

THEOREM 3.2.12. *The orthogonal decomposition*

$$\boxed{\mathtt{Pol}_{\mathbf{K};d}(E) = \mathtt{Harm}_{\mathbf{K}}(E;0) \bigoplus \mathtt{Harm}_{\mathbf{K}}(E;2) \bigoplus \ldots \bigoplus \mathtt{Harm}_{\mathbf{K}}(E;d)} \tag{3.117}$$

holds.

Proof. The decomposition (3.117) follows from Theorem 3.2.10 and from (3.116) taking into account that

$$\boxed{\mathtt{Harm}_{\mathbf{K}}(E;2k) \subset \mathtt{Harm}(E;2k) \equiv \mathtt{Harm}_{\mathbf{R}}(E_{\mathbf{R}};2k)} \tag{3.118}$$

so, the subspaces $\mathtt{Harm}_{\mathbf{K}}(E;2k),\ 0 \le k \le \frac{d}{2}$, are orthogona according to (3.83). □

3.3 The Projective Addition Theorem

The inner product (3.54) is defined, in particular, for the polynomial functions on the projective space. With any $k \in \mathbf{N}$ the space $\mathtt{Harm}_{\mathbf{K}}(E;2k)$ is invariant with respect to the natural unitary representation of the unitary group $U(E) \subset O(E_{\mathbf{R}})$, see 3.61. Let

$$\boxed{\hbar_{m,2k} = \dim \mathtt{Harm}_{\mathbf{K}}(E;2k)} \tag{3.119}$$

and let $(s_{ki})_{i=1}^{\hbar_{m,2k}}$ be an orthonormal basis in $\mathtt{Harm}_{\mathbf{K}}(E;2k)$. Like (3.62), we introduce the function

$$f(x,y) = \sum_{i=1}^{\hbar_{m,2k}} \overline{s_{ki}(x)} s_{ki}(y) \quad (x,y \in \mathbf{S}(E)). \tag{3.120}$$

By the same argument as in Section 3.1 this function is independent of the basis, in particular, this is unitary invariant. By Theorem 1.9.32 (the Witt Theorem) $f(x,y)$ only depends on $\langle x,y\rangle$. Moreover,

$$f(x\gamma, y) = f(x,y), \quad \gamma \in U(\mathbf{K}). \tag{3.121}$$

Hence, $f(x,y)$ actually is a function of $|\langle x,y\rangle|$ or, equivalently, of $|\langle x,y\rangle|^2$,

$$f(x,y) = q_{m,k}(|\langle x,y\rangle|^2) \tag{3.122}$$

where $q_{m,k}$ is a function $[0,1] \to \mathbf{C}$. Thus,

$$\boxed{\sum_{i=1}^{\hbar_{m,2k}} \overline{s_{ki}(x)} s_{ki}(y) = q_{m,k}(|\langle x,y\rangle|^2)} \tag{3.123}$$

LEMMA 3.3.1. *$q_{m,k}$ is a polynomial of degree $\le k$.*

The proof below is a slight modification of the proof of Lemma 3.1.16.

Proof. Choose a pair $x_0, y_0 \in \mathbf{S}(E)$ such that $x_0 \perp y_0$ and consider

$$x(\vartheta) = y_0 \cos\vartheta + x_0 \sin\vartheta, \quad 0 \le \vartheta < 2\pi.$$

Then

$$|\langle x(\vartheta), y_0\rangle|^2 = \cos^2\vartheta. \tag{3.124}$$

By the lifting $\mathtt{Harm}_{\mathbf{K}}(E;2k) \to \mathcal{H}_{\mathbf{K}}(E;2k)$ we get

$$s_{ki}(x(\vartheta)) = s_{ki}^{(h)}(y_0 + x_0\tan\vartheta)\cos^{2k}\vartheta = \sum_{j=0}^{2k} b_{ki,j}\sin^{2k-j}\vartheta \cdot \cos^j\vartheta \tag{3.125}$$

where $b_{ki,j}$ are some complex coefficients. By (3.123), (3.124) and (3.125)

$$q_{m,k}(\cos^2\vartheta) = \sum_{j=0}^{2k} a_{kj}\sin^{2k-j}\vartheta \cdot \cos^j\vartheta$$

with some new coefficients a_{kj}. By substitution $\vartheta \mapsto \pi - \vartheta$,

$$q_{m,k}(\cos^2\vartheta) = \sum_{j=0}^{2k}(-1)^j a_{kj}\sin^{2k-j}\vartheta \cdot \cos^j\vartheta.$$

Hence, $(-1)^j a_{kj} = a_{kj}$, i.e. $a_{kj} = 0$ for odd j. Thus,

$$q_{m,k}(u) = \sum_{j=0}^{k} a_{k,2j}(1-u)^{k-j}u^j, \quad 0 \le u \le 1.$$

□

The final explicit expression for $q_{m,k}(u)$ (see below) depends on the basis field $\mathbf{K}$.

THEOREM 3.3.2. *For any orthonormal basis $(s_{kj})_{j=1}^{\hbar_{m,2k}}$ in the space $\mathtt{Harm}_{\mathbf{K}}(E;2k)$ of $U(\mathbf{K})$-invariant spherical harmonics of degree $2k$ the* **Projective Addition Formula**

$$\boxed{\sum_{j=1}^{\hbar_{m,2k}} \overline{s_{kj}(x)}s_{kj}(y) = b_{m,k}^{(\delta)}P_k^{(\alpha,\beta)}(2|\langle x,y\rangle|^2 - 1)} \tag{3.126}$$

holds with the coefficient

$$\boxed{b_{m,k}^{(\delta)} = \frac{\hbar_{m,2k}}{P_k^{(\alpha,\beta)}(1)} = \frac{P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|^2_{\Omega_{\alpha,\beta}}}}, \tag{3.127}$$

where

$$\alpha = \frac{\delta m - \delta - 2}{2}, \quad \beta = \frac{\delta - 2}{2}, \quad \delta = [\mathbf{K} : \mathbf{R}]. \tag{3.128}$$

Proof. In contrast to the spherical case we can start with the orthogonality relation

$$\mathtt{Harm}_{\mathbf{K}}(E,2k) \perp \mathtt{Harm}_{\mathbf{K}}(E,2l), \quad k \neq l.$$

For this reason we can conclude that

$$\iint q_{m,k}(|\langle x,y\rangle|^2) q_{m,l}(|\langle x,y\rangle|^2)\, d\sigma(x)\, d\sigma(y) = 0, \quad k \neq l \tag{3.129}$$

from the identity

$$\iint q_{m,k}(|\langle x,y\rangle|^2) q_{m,l}(|\langle x,y\rangle|^2)\, d\sigma(x)\, d\sigma(y) = \sum_{j=1}^{\hbar_{m,2k}} \sum_{i=1}^{\hbar_{m,2l}} |(s_{kj}, s_{li})|^2 \tag{3.130}$$

which is quite similar to (3.82). On the other hand, by the integration formula (2.33)

$$\int q_{m,k}(|\langle x,y\rangle|^2) q_{m,l}(|\langle x,y\rangle|^2)\, d\sigma(y) = \int_{-1}^{1} q_{m,k}\left(\frac{1+v}{2}\right) q_{m,l}\left(\frac{1+v}{2}\right) \Omega_{\alpha,\beta}(v)\, dv \tag{3.131}$$

with α and β given by (2.34), the same as in (3.128). By the next integration

$$\iint q_{m,k}(|\langle x,y\rangle|^2) q_{m,l}(|\langle x,y\rangle|^2)\, d\sigma(x)\, d\sigma(y) = \int_{-1}^{1} q_{m,k}\left(\frac{1+v}{2}\right) q_{m,l}\left(\frac{1+v}{2}\right) \Omega_{\alpha,\beta}(v)\, dv. \tag{3.132}$$

Comparing (3.132) to (3.129) we get

$$\int_{-1}^{1} q_{m,k}\left(\frac{1+v}{2}\right) q_{m,l}\left(\frac{1+v}{2}\right) \Omega_{\alpha,\beta}(v)\, dv = 0, \quad k \neq l. \tag{3.133}$$

Thus, the polynomials $q_{m,k}\left(\frac{1+v}{2}\right)$ are orthogonal with the weight $\Omega_{\alpha,\beta}$ on $[-1,1]$, $\deg q_{m,k} \leq k$. Hence,

$$q_{m,k}\left(\frac{1+v}{2}\right) = b_{m,k}^{(\delta)} P_k^{(\alpha,\beta)}(v) \tag{3.134}$$

where $b_{m,k}^{(\delta)}$ is a constant coefficient. Equivalently,

$$q_{m,k}(u) = b_{m,k}^{(\delta)} P_k^{(\alpha,\beta)}(2u-1) \tag{3.135}$$

and now (3.123) turns into (3.126). It remains to determine the coefficient $b_{m,k}^{(\delta)}$ in (3.126).

Setting $y = x$ in (3.126) we get

$$\sum_{j=1}^{\hbar_{m,2k}} |s_{kj}(x)|^2 = q_{m,k}(1) = b_{m,k}^{(\delta)} P_k^{(\alpha,\beta)}(1). \tag{3.136}$$

Integrating (3.136) over $\mathbf{S}(E)$ we get

$$\hbar_{m,2k} = q_{m,k}(1) = b_{m,k}^{(\delta)} P_k^{(\alpha,\beta)}(1). \tag{3.137}$$

We already have the first equality (3.127). However, in contrast to the spherical case we do not know the dimensional quantity $\hbar_{m,2k}$. The latter can be calculated by comparison of two expressions from (3.127) but the second one must be obtained before.

It follows from (3.131) that

$$\int_{-1}^{1} q_{m,k}^2\left(\frac{1+v}{2}\right)\Omega_{\alpha,\beta}(v)\,dv = \int q_{m,k}^2(|\langle x,y\rangle|^2)\,d\sigma(y) = \sum_{j=1}^{\hbar_{m,2k}} |s_{kj}(x)|^2 = q_{m,k}(1). \tag{3.138}$$

Hence,

$$\left(b_{m,k}^{(\delta)}\right)^2 \int_{-1}^{1} \left(P_k^{(\alpha,\beta)}(v)\right)^2 \Omega_{\alpha,\beta}(v)\,dv = b_{m,k}^{(\delta)} P_k^{\alpha,\beta}(1).$$

This immediately yields the second expression for $b_{m,k}^{(\delta)}$ in (3.127). (Note that $b_{m,k}^{(\delta)} \neq 0$ according to (3.137).) □

COROLLARY 3.3.3. *Every polynomial $q_{m,k}$ is of degree k,*

$$\boxed{\deg q_{m,k} = k}\ . \tag{3.139}$$

In particular,

$$\boxed{q_{m,0}(u) = \hbar_{m,0} = 1}\ . \tag{3.140}$$

In general, we have

COROLLARY 3.3.4.

$$\boxed{\hbar_{m,2k} = \left(\frac{P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|_{\Omega_{\alpha,\beta}}}\right)^2} \tag{3.141}$$

In turn, this yields

COROLLARY 3.3.5. *The following* **dimension formulae** *are valid*

$$\boxed{\dim \mathtt{Pol}_{\mathbf{R}}(E;2t) = \binom{m+2t-1}{m-1}} \tag{3.142}$$

and

$$\boxed{\dim \mathtt{Pol}_{\mathbf{C}}(E;2t) = \binom{m+t-1}{m-1}^2}\ , \tag{3.143}$$

and

$$\boxed{\dim \mathtt{Pol}_{\mathbf{H}}(E;2t) = \frac{1}{2m-1}\binom{2m+t-2}{2m-2}\cdot\binom{2m+t-1}{2m-2}} \quad . \tag{3.144}$$

Proof. By Theorem 3.2.12 and Corollary 3.3.4

$$\dim \mathtt{Pol}_{\mathbf{K}}(E;2t) = \sum_{k=0}^{t} \hbar_{m,2k} = \sum_{k=0}^{t}\left(\frac{P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|_{\Omega_{\alpha,\beta}}}\right)^2 .$$

The latter expression is just $\Lambda_{\mathbf{K}}(m;2t)$ introduced in (2.19), so that its final form follows from Theorem 2.1.1. □

Thus, the unified form of Corollary 3.3.5 is

$$\boxed{\dim \mathtt{Pol}_{\mathbf{K}}(E;2t) = \Lambda_{\mathbf{K}}(m,2t)} \quad . \tag{3.145}$$

Now we formulate the projective counterpart of Lemma 3.1.22. The proof can be done similarly but, of course, on the base of the Projective Addition Theorem instead of the spherical one. Here we have to consider a polynomial

$$f(v) = \sum_{k=0}^{\deg f} c_{m,k}^{(\delta)}(f) q_{m,k}\left(\frac{1+v}{2}\right) \tag{3.146}$$

where

$$c_{m,k}^{(\delta)}(f) = \frac{\displaystyle\int_{-1}^{1} q_{m,k}\left(\frac{1+v}{2}\right) f(u)\Omega_{\alpha,\beta}(u)\,dv}{\displaystyle\int_{-1}^{1} q_{m,k}^2\left(\frac{1+v}{2}\right)\Omega_{\alpha,\beta}(u)\,dv}. \tag{3.147}$$

In fact, the integral in the denominator is equal to $q_{m,k}(1)$ by (3.138) and it is also $\hbar_{m,2k}$ by (3.137). Thus,

$$c_{m,k}^{(\delta)}(f) = \frac{1}{q_{m,k}(1)}\int_{-1}^{1} q_{m,k}\left(\frac{1+v}{2}\right) f(u)\Omega_{\alpha,\beta}(u)\,dv. \tag{3.148}$$

In particular,

$$c_{m,0}^{(\delta)}(f) = \int_{-1}^{1} f(u)\Omega_{\alpha,\beta}(u)\,dv. \tag{3.149}$$

LEMMA 3.3.6. *For any finite subset $X \subset \mathbf{S}(E)$ and for any function $\lambda : X \to \mathbf{C}$ the equality*

$$\boxed{\sum_{x,y\in X} f(2|\langle x,y\rangle|^2 - 1)\overline{\lambda(x)}\lambda(y) = \sum_{k=0}^{\deg f} c_{m,k}^{(\delta)}(f) \sum_{i=1}^{\hbar_{m,2k}} \left|\sum_{x\in X} s_{ki}(x)\lambda(x)\right|^2} \tag{3.150}$$

holds.

The last (but not least) consequence of the Projective Addition Theorem is the following **Completeness Theorem.**

THEOREM 3.3.7. *In the space* $\mathtt{Pol}_{\mathbf{K}}(E;2t)$ *the system of elementary polynomial functions of degree* $2t$ *is complete,*

$$\boxed{\mathtt{Pol}_{\mathbf{K}}(E;2t) = \mathrm{Span}\{|\langle\cdot,y\rangle|^{2t} : y\in E\}} \quad . \tag{3.151}$$

Proof. By Corollary 1.9.18 it is sufficient to prove that if $\phi\in\mathtt{Pol}_{\mathbf{K}}(E;2t)$ is such that

$$\int |\langle x,y\rangle|^{2t}\psi(x)\,d\sigma(x) = 0 \tag{3.152}$$

for all $y\in E$ then $\psi = 0$. Condition (3.152) can be rewritten as

$$\int |\langle x,y\rangle|^{2t}\,d\tau(x) = 0, \quad y\in E, \tag{3.153}$$

where $d\tau = \phi d\sigma$. Applying the Laplacian Δ in variable y to (3.153) and using Lemma 2.3.5 we reduce the exponent $2t$ to $2t-2$, i.e.

$$\int |\langle x,y\rangle|^{2t-2}\,d\tau(x) = 0$$

which means that

$$\int |\langle x,y\rangle|^{2t-2}\phi(x)\,d\sigma(x) = 0.$$

By iteration of this procedure we get

$$\int |\langle x,y\rangle|^{2k}\phi(x)\,d\sigma(x) = 0 \tag{3.154}$$

for $0\le k\le t$. By Lemma 3.3.1

$$\int q_{m,k}(|\langle x,y\rangle|^2)\phi(x)\,d\sigma(x) = 0.$$

The latter can be rewritten as

$$\sum_{j=1}^{\hbar_{m,2k}} (s_{kj},\phi)s_{kj}(y) = 0$$

by the Projective Addition Theorem. Therefore,

$$(s_{kj},\phi) = 0, \quad 1\le j\le \hbar_{m,2k} \tag{3.155}$$

since the system $(s_{kj})_{j=1}^{\hbar_{m,2k}}$ is linearly independent. However, the system

$$\bigcup\{s_{kj}:\ 1\le j\le \hbar_{m,2k},\quad 0\le k\le t\}$$

is a basis in $\mathtt{Pol}_{\mathbf{K}}(E, 2t)$ by Theorem 3.2.12 and Corollary 3.2.8. The equalities (3.155) say that ϕ is orthogonal to all elements of this basis. By Proposition 1.9.2 $\phi \perp \mathtt{Pol}_{\mathbf{K}}(E; 2t)$ so, $\phi = 0$ being an element of the same space. □

In fact, the parametric space E in (3.151) can be restricted to the sphere $\mathbf{S}(E)$ and even to the projective space $\mathbf{P}(E)$ realized as a "fundamental domain" on the sphere. Thus,

$$\boxed{\mathtt{Pol}_{\mathbf{K}}(E; 2t) = \mathrm{Span}\{|\langle \cdot, y\rangle|^{2t} : y \in \mathbf{S}(E)\}} \quad . \tag{3.156}$$

Chapter 4

Cubature formulas and isometric embeddings $\ell_2^m \to \ell_p^n$

4.1 Spherical codes, cubature formulas and designs

`In this Section` E `is a real` m`-dimensional Euclidean space.`

A **spherical code** is a nonempty finite subset $X \subset \mathbf{S}(E)$. Its **angle set** is defined as

$$\boxed{A(X) = \{\langle x, y\rangle : \; x, y \in X, \quad x \neq y\} \subset [-1, 1)} \; . \tag{4.1}$$

In terms of the Coding Theory the set $\arccos A(X)$ is the **distance set** (see Section 1.10). The following result is purely combinatoric because X is arbitrary. Lemma 3.1.22 is the main key to it.

THEOREM 4.1.1. *Let X be a spherical code, $|X| = n$, $|A(X)| = s$. Then*

$$\boxed{n \leq \binom{m+s-2}{m-1} + \binom{m+s-1}{m-1}} \; . \tag{4.2}$$

Proof. Consider the polynomial F, $\deg F = s$, such that $F|A(X) = 0$. Then $F(1) \neq 0$ since $1 \notin A(X)$. Now we apply Lemma 3.1.22. The identity (3.99) turns into

$$F(1) \sum_{x \in X} |\lambda(x)|^2 = \sum_{k=0}^{s} c_{m,k}(F) \sum_{i=1}^{h_{m,k}} |\mu_{ik}(x)|^2 \tag{4.3}$$

where

$$\mu_{ik} = \sum_{x \in X} S_{ki}(x)\lambda(x)$$

are linear forms of the vectors $[\lambda(x)]_{x\in X}$. The left expression in(4.3) shows that the rank of the Hermitian quadratic form (3.98) is equal to n. The right expression shows that the same rank does not exceed the total number of summands there,

$$n \le \sum_{k=0}^{s} h_{m,k} \ . \tag{4.4}$$

By (3.31)

$$\boxed{n \le \dim \mathtt{Pol}_s(E)} \ . \tag{4.5}$$

By (3.39) the latter is just the upper bound (4.2). □

Now we pass to the special spherical codes which are so nicely arranged on the sphere that the integration over the sphere is equivalent to a weighted averaging over a code. Certainly, this is possible only for finite-dimensional spaces of integrands, say, for $\mathtt{Pol}(E;d)$, $\mathtt{Pol}_d(E)$, etc., but with an upper bound for d.

DEFINITION 4.1.2. *A* **spherical cubature formula of index** d *is an identity*

$$\int \phi \, d\varrho = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}(E;d), \tag{4.6}$$

where σ is the normalized Lebesgue measure on the sphere $\mathbf{S}(E)$ and ϱ is a normalized finitely supported measure,

$$\mathrm{supp}\varrho = \{x_k\}_1^n \subset \mathbf{S}(E),$$

and

$$\int \phi \, d\varrho \equiv \int_{\mathrm{supp}\varrho} \phi \, d\varrho = \sum_{x\in \mathrm{supp}\varrho} \phi(x)\varrho(x) \equiv \sum_{x} \phi(x)\varrho(x). \tag{4.7}$$

The points $x_1, \dots, x_n$ are called the **nodes** *and its measures $\varrho_k = \varrho(x_k)$, $1 \le k \le n$, are called the* **weights**. *The set* suppϱ *is called the* **support of the spherical cubature formula.**

REMARK 4.1.3. *Formula (4.6) with even d automatically implies that ϱ is normalized since $\mathbf{1} \in \mathtt{Pol}(E;d)$ in this case.*

The identity (4.6) can be also rewritten as

$$\sum_{k=1}^{n} \phi(x_k)\varrho_k = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}(E;d), \tag{4.8}$$

where $\varrho_k > 0$, $1 \le k \le n$.

A spherical cubature formula of index d in the case of equal weights, i.e with

$$\varrho_1 = \ldots = \varrho_n = \frac{1}{n}, \tag{4.9}$$

is called a **Chebyshev type cubature formula**. Its support is called a **spherical design of index** d, a term from the Algebraic Combinatorics. Thus, the left integral in (4.6) is equal to the arithmetic mean of ϕ over any spherical design X of index d,

$$\frac{1}{|X|}\sum_{x\in X}\phi(x)=\int\phi\,d\sigma,\quad \phi\in\texttt{Pol}(E;d). \tag{4.10}$$

In the Algebraic Combinatorics the spherical cubature formulas are also called the **weighted spherical designs**.

PROPOSITION 4.1.4. *A spherical code X is a spherical design of even index d if and only if*

$$\sum_{x\in X}\phi(gx)=\sum_{x\in X}\phi(x),\qquad \phi\in\texttt{Pol}(E;d), \tag{4.11}$$

for all $g\in O(E)$.

Proof. (4.10)⇒(4.11). This immediately follows from the unitary invariance of the measure σ.

(4.11)⇒(4.10). The identity (4.11) means that the linear functional

$$\phi\mapsto\frac{1}{|X|}\sum_{x\in X}\phi(x) \tag{4.12}$$

is orthogonally invariant. On the other hand, this functional can be represented in the Riesz form,

$$\phi\mapsto\int\overline{\psi}\phi\,d\sigma,$$

where $\psi\in\texttt{Pol}(E;d)$ and ψ is uniquely determined. Hence, the function ψ must be orthogonally invariant. Since the action of the orthogonal group on $\mathbf{S}(E)$ is transitive, $\psi(x)=\text{const}$. This constant is 1 since $\mathbf{1}\to 1$ in (4.12). □

Note that if d is odd (i.e. $\varepsilon_d=1$) then

$$\int\phi\,d\sigma=0,\quad \phi\in\texttt{Pol}(E;d),$$

so, (4.6) becomes

$$\int\phi\,d\varrho=\sum_x\phi(x)\varrho(x)=0,\quad \phi\in\texttt{Pol}(E;d). \tag{4.13}$$

In the case of a spherical design (4.13) becomes

$$\sum_x\phi(x)=0,\quad \phi\in\texttt{Pol}(E;d). \tag{4.14}$$

A spherical code X is called **antipodal** if $X=-X$, i.e. with every point $x\in X$ the opposite point $-x$ also belongs to X. Obviously, *any antipodal spherical code is a spherical design of index d for any odd d.* However, it is easy to construct a spherical design of index 1 which is not antipodal. Indeed, for $d=1$ (4.14) is equivalent to

$$\sum_{x\in X}x=0 \tag{4.15}$$

which just means that the baricenter of the set X is at the origin. Obviously, (4.15) includes not only antipodal spherical codes.

A spherical code X is called **podal** if $X \cap (-X) = \emptyset$, i.e. for every $x \in X$ its opposite point $-x$ does not belong to X.

PROPOSITION 4.1.5. *A spherical cubature formula of index d is equivalent to the system of equalities*

$$\int \phi \, d\varrho = \sum_x \phi(x)\varrho(x) = 0, \quad \phi \in \mathtt{Harm}(E;k), \quad k \in \mathcal{E}_d \setminus \{0\}, \tag{4.16}$$

and, in addition,

$$\int d\varrho = \sum_x \varrho(x) = 1 \tag{4.17}$$

in the case of even d .

Proof. The cubature formula (4.6) implies (4.16) since $\phi \in \mathtt{Harm}(E;k)$ so, $\phi \in \mathtt{Pol}(E;d)$ for $k \in \mathcal{E}_d$ according to (3.50). Thus, (4.6) is applicable to $\phi \in \mathtt{Harm}(E;k), \quad k \in \mathcal{E}_d$. It remains to note that $\int \phi \, d\sigma = 0$ by (3.60). In addition, if d is even then (4.6)⇒(4.17).

Conversely, the harmonic decomposition (3.50),

$$\phi = \sum_{k \in \mathcal{E}_d} \phi_k$$

yields

$$\int \phi \, d\sigma = \sum_{k \in \mathcal{E}_d} \int \phi_k \, d\sigma = \int \phi_{\varepsilon_d} \, d\sigma. \tag{4.18}$$

The integral (4.18) is ϕ_0 if $\varepsilon_d = 0$ since $\phi_0 = \text{const}$ and σ is normalized. If $\varepsilon_d = 1$ then the integral (4.18) is equal to zero.

Similarly,

$$\int \phi \, d\varrho = \int \phi_{\varepsilon_d} \, d\varrho \tag{4.19}$$

by (4.16). The values of integrals (4.19) and (4.18) coincide since in the case $\varepsilon_d = 0$ the measure ϱ is normalized and, moreover, $\int \phi_1 \, d\rho = 0$.□

COROLLARY 4.1.6. *A spherical code X is a spherical design of index d if and only if*

$$\sum_{x \in X} \phi(x) = 0, \quad \phi \in \mathtt{Harm}(E;k), \quad k \in \mathcal{E}_d, \; k \geq 1. \tag{4.20}$$

Indeed, (4.17) is automatically valid for the spherical designs.

COROLLARY 4.1.7. *A spherical cubature formula of index d is a spherical cubature formula of every index $k \in \mathcal{E}_d$.*

DEFINITION 4.1.8. *A* **spherical cubature formula of degree** d *(or* **strength** d*) is a spherical cubature formula of all indices* $2t, 2t-2, \ldots, 0$, *i.e.*

$$\int \phi \, d\varrho = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}(E;k), \quad 0 \leq k \leq d. \tag{4.21}$$

In view of the homogeneous lifting (3.9) this definition is equivalent to

$$\int \phi \, d\varrho = \int \phi \, d\sigma, \quad \phi \in \texttt{Pol}_d(E). \tag{4.22}$$

Therefore, by Corollary 3.1.13 we have

PROPOSITION 4.1.9. *A spherical cubature formula of degree d is the same as a spherical cubature formula of indices $k = d-1,\ d$.*

COROLLARY 4.1.10. *Any antipodal spherical cubature formula of even index d has also degree d.*

COROLLARY 4.1.11. *For any spherical cubature formula of even index d its* **symmetrization**

$$X \mapsto \widetilde{X} = X \cup (-X), \quad \varrho \mapsto \widetilde{\varrho}, \quad \widetilde{\varrho}(\pm x) = \frac{1}{2}\varrho(x) \quad (x \in X), \tag{4.23}$$

yields an antipodal spherical cubature formula of degree d.

Obviously, $|\widetilde{X}| \le 2|X|$ and $|\widetilde{X}| = |X|$ if and only if X is podal.

In particular, the symmetrization (4.23) generates a correspondence between podal and antipodal spherical codes such that each antipodal spherical code can be obtained from exactly $2^{|X|}$ podal codes.

Under Definition 4.1.8 the relevant version of Proposition 4.1.5 is

PROPOSITION 4.1.12. *A spherical cubature formula of degree d is equivalent to the system of equalities*

$$\int \phi \, d\varrho = \sum_x \phi(x)\varrho(x) = 0, \quad \phi \in \texttt{Harm}(E;k), \quad 1 \le k \le d. \tag{4.24}$$

COROLLARY 4.1.13. *A spherical code X is a spherical design of degree d if and only if*

$$\sum_{x \in X} \phi(x) = 0, \quad \phi \in \texttt{Harm}(E;k), \quad 1 \le k \le d. \tag{4.25}$$

A spherical design of degree d is called a **d-design.**

Now let us note that the upper bound (4.2) is valid for any cubature formula as a spherical code with special properties. Below we establish a lower bound for the number

$$n = |X| = |\mathrm{supp}\varrho| \tag{4.26}$$

of nodes of a spherical cubature formula of even index d. *If d is odd then there is no nontrivial lower bound for n.* Indeed, any pair of mutually opposite points on the sphere is a spherical design of index d.

THEOREM 4.1.14. *For any spherical cubature formula of even index d the inequality*

$$\boxed{n \ge \binom{m + \frac{d}{2} - 1}{m-1}} \tag{4.27}$$

holds.

The binomial coefficient in (4.27) is just $\dim \mathtt{Pol}(E; \frac{d}{2})$, see (3.49) and (3.38). Thus, one can rewrite (4.27) as

$$\boxed{n \geq \dim \mathtt{Pol}(E; \tfrac{d}{2})} \;. \tag{4.28}$$

Proof. Suppose to the contrary and then consider the interpolation problem for $\theta \in \mathtt{Pol}(E; \frac{d}{2})$:

$$\theta(x_k) = 0, \quad 1 \leq k \leq n.$$

This problem has a nontrivial solution. By the given cubature formula for the function $\phi = |\theta|^2 \in \mathtt{Pol}(E; d)$ we get

$$\int |\theta|^2 \, d\sigma = 0,$$

whence $\theta = 0$, the contradiction. □

The problem arises whether the lower bound (4.27) is exact and, in the case of affirmative answer, what is the corresponding cubature formula.

DEFINITION 4.1.15. *A spherical cubature formula of even index d is called* **tight** *if in (4.27) the equality is attained.*

It is easy to see that *the support X of any tight spherical cubature formula is podal.* Indeed, if $x_0 \in X \cap (-X)$ then one of nodes x_0 or $-x_0$ can be omitted in (4.7) since $\phi(-x_0) = \phi(x_0)$. Thus, the number of nodes becomes less than lower bound (4.27).

A spherical code $X = (x_k)_1^n$ is called **t-interpolating** if for every vector $[\zeta_k]_1^n \in \mathbf{C}^n$ there exists a unique form $\phi \in \mathtt{Pol}(E; t)$ such that $\phi(x_k) = \zeta_k, \; 1 \leq k \leq n$, cf. [41].

LEMMA 4.1.16. *A spherical code $X = (x_k)_1^n$ is t-interpolating if and only if*

$$n = \binom{m+t-1}{m-1} \tag{4.29}$$

and there is no a nontrivial form $\phi \in \mathtt{Pol}(E; t)$ such that $\phi | X = 0$.

Proof. The t-interpolating property means that the mapping $\phi \mapsto (\phi(\zeta_k))_1^n$ is an isomorphism $\mathtt{Pol}(E; t) \to \mathbf{C}^n$. Equivalently, the same mapping is injective and $n = \dim \mathtt{Pol}(E; t)$. □

Suppose that $X = (x_k)_1^n$ is a t-interpolating system. The basis $(L_j)_1^n$ of the space $\mathtt{Pol}(E; t)$ defined by the conditions

$$L_j(x_k) = \delta_{jk}, \quad 1 \leq j, k \leq n, \tag{4.30}$$

is called the **Lagrange basis** corresponding to X.

THEOREM 4.1.17. [36]. *If a spherical cubature formula of index $d = 2t$ is tight then*

(i) *its support $(x_k)_1^n$ is a t-interpolating system;*

(ii) *the corresponding Lagrange basis $(L_j)_1^n$ is orthogonal, i.e.*

$$(L_j, L_k) = \int \overline{L_j} L_k \, d\sigma = 0, \quad j \neq k; \tag{4.31}$$

(iii) *the weights are*

$$\varrho_j = \|L_j\|^2 = \int |L_j|^2 \, d\sigma, \;\; 1 \le j \le n. \tag{4.32}$$

Conversely, let a spherical code $(x_k)_1^n$ *be a* t*-interpolating system with property* (**ii**). *Then* $(x_k)_1^n$ *is the support of the tight spherical cubature formula of index* $d = 2t$ *with weights (4.32).*

Proof. Let the formula be tight. Then for any form $\phi \in \mathtt{Pol}(E;t)$ we have

$$\int |\phi|^2 d\sigma = \sum_{k=1}^{n} |\phi(x_k)|^2 \varrho_k. \tag{4.33}$$

If $\phi(x_k) = 0, \; 1 \le k \le n$, then $\phi = 0$, i.e. the mapping $\phi \mapsto (\phi(x_k))_1^n$ is injective. Moreover, $n = \dim \mathtt{Pol}(E;t)$. As a result, we get **(i)** by Lemma 4.1.16. Using the cubature formula with $\phi = \overline{L_j} L_k$ or $|L_k|^2$ we get **(ii)** and **(iii)** by (4.30).

Conversely, any form $\phi \in \mathtt{Pol}(E;d)$ is a linear combination of the products $\overline{L_j} L_k \in \mathtt{Pol}(E;d)$,

$$\phi = \sum_{j,k=1}^{n} b_{jk} \overline{L_j} L_k \tag{4.34}$$

with some coefficients b_{jk}. (Indeed, any monomial of degree d is product of two monomials of degree t and the latter can be both decomposed for a basis (L_j).) Hence,

$$\sum_{l=1}^{n} \phi(x_l) \varrho_l = \sum_{j,k,l=1}^{n} b_{jk} \overline{L_j(x_l)} L_k(x_l) \varrho_l = \sum_{k=1}^{n} b_{kk} \varrho_k. \tag{4.35}$$

On the other hand,

$$\int \phi \, d\sigma = \sum_{j,k=1}^{n} b_{jk} \int \overline{L_j} L_k \, d\sigma = \sum_{k=1}^{n} b_{kk} \varrho_k. \tag{4.36}$$

So, the cubature formula under consideration is of index d. The formula is tight by Lemma 4.1.16.□

Below we need the following statement of a general nature.

LEMMA 4.1.18. *For a fixed pole* $x \in \mathbf{S}(E)$ *the evaluation functional* $\phi \mapsto \phi(x)$ *on* $\mathtt{Pol}(E;t)$ *can be represented as*

$$\phi(x) = \int \overline{\theta_x(y)} \phi(y) \, d\sigma(y) \tag{4.37}$$

where $\theta_y \in \mathtt{Pol}(E;t)$. *The norm* $\|\theta_x\|$ *does not depend on* y.

Therefore, $\|\theta_x\|$ only depends on m and t.

Proof. The first statement is actually the Riesz representation of the evaluation functional. Further, (4.37) implies

$$\int \overline{\theta_{gx}(y)} \phi(y) \, d\sigma(y) = \phi(gx) = \int \overline{\theta_x(y)} \phi(gy) \, d\sigma(y), \quad g \in O(E). \tag{4.38}$$

Taking $\phi(y) = \theta_{gx}(y)$ we obtain

$$\|\theta_{gx}\|^2 \le \|\theta_x\|^2 \int |\theta_{gx}(gy)|^2 \, d\sigma(y)$$

by the Schwartz inequality. Since the measure σ is orthogonally invariant, the latter integral is equal to $\|\theta_{gx}\|$ so, the inequality takes the form $\|\theta_{gx}\| \le \|\theta_x\|$ for all $y \in \mathbf{S}(E)$ and all $g \in O(E)$. Changing g for g^{-1} and x for gx we obtain the converse inequality. Thus, $\|\theta_{gx}\| = \|\theta_x\|$ and it remains to recall that $O(E)$ acts on $\mathbf{S}(E)$ transitively. □

Combining this lemma with Theorem 4.1.17 we obtain

COROLLARY 4.1.19. *In any tight spherical cubature formula the weights are equal.*

In other words, *the support of any tight spherical cubature formula of index d is a spherical design of index d (a* **tight spherical design***)*. As we already know, this spherical design is podal.

Proof. Since the basis $(L_j) \subset \mathtt{Pol}(E;t)$ is orthogonal, we have the decomposition

$$\theta_x = \sum_{j=1}^{n} \frac{(\theta_x, L_j)}{\|L_j\|^2} L_j,$$

whence

$$\theta_x(y) = \sum_{j=1}^{n} \frac{L_j(x)L_j(y)}{\varrho_j}$$

by (4.37) and (4.32). In particular,

$$\theta_{x_k}(y) = \frac{1}{\varrho_k} L_k(y), \quad 1 \le k \le n, \tag{4.39}$$

since (L_k) is the Lagrange basis. Passing to the norms in (4.39) we obtain

$$\|\theta_{x_k}\| = \frac{\|L_k\|}{\varrho_k} = \frac{1}{\sqrt{\varrho_k}}.$$

It remains to apply Lemma 4.1.18 again. □

REMARK 4.1.20. As follows from the proof

$$\boxed{\|\theta_x\|^2 = n = \binom{m+t-1}{m-1}} \tag{4.40}$$

because of (4.9). □

4.2 The linear programming approach

This approach to the lower bounds is based on the following

LEMMA 4.2.1. *Any spherical cubature formula of index d is equivalent to the identity*

$$\sum_{x,y} F(\langle x, y\rangle)\varrho(x)\varrho(y) = c_{m,0}(F) \equiv \int_{-1}^{1} F(u)\Omega_m(u)\,du, \quad F \in \Pi_d^{(\varepsilon_d)}, \tag{4.41}$$

where $\Omega_m(u)$ is the normalized weight (2.10).

Recall that ε_d is the residue of d modulo 2 and, according to (3.88), $\Pi_d^{(\varepsilon_d)}$ is the space of complex polynomials of degrees $\leq d$ which are even for even d and odd for odd d. Note that $c_{m,0}(F) = 0$ if d is odd.

Proof. Let $F \in \Pi_d^{(\varepsilon_d)}$. Then the Fourier coefficients $c_{m,k}(F)$ are equal to zero for $k \notin \mathcal{E}_d$, $0 \leq k \leq d$. The identity (3.99) with $\lambda = \varrho$ is

$$\sum_{x,y} F(\langle x, y\rangle)\varrho(x)\varrho(y) = c_{m,0}(F)\left(\sum_x \varrho(x)\right)^2 + \sum_{k\in\mathcal{E}_d\setminus\{0\}} c_{m,k}(F)\sum_{i=1}^{h_{m,k}}\Big|\sum_x S_{ki}(x)\varrho(x)\Big|^2. \tag{4.42}$$

If ϱ comes from a cubature formula of index d then (4.42) reduces to (4.41) because of Proposition 4.1.5.

Conversely, let (4.41) takes place. Then (4.42) turns into

$$c_{m,0}(F) = c_{m,0}(F)\left(\sum_x \varrho(x)\right)^2 + \sum_{k\in\mathcal{E}_d\setminus\{0\}} c_{m,k}(F)\sum_{i=1}^{h_{m,k}}\Big|\sum_x S_{ki}(x)\varrho(x)\Big|^2.$$

The coefficients $c_{m,k}(F)$ with $k \in \mathcal{E}_d \setminus \{0\}$ are independent parameters when F runs over $\Pi_d^{(\varepsilon_d)}$. Moreover, $c_{m,0}(F)$ can be included in this system of parameters if d is even. Hence,

$$\sum_{i=1}^{h_{m,k}}\Big|\sum_x S_{ki}(x)\varrho(x)\Big|^2 = 0, \quad k \in \mathcal{E}_d \setminus \{0\}, \tag{4.43}$$

and, in addition,

$$\sum_x \varrho(x) = 1$$

if d is even. It follows from (4.43) that

$$\sum_x S_{ki}(x)\varrho(x) = 0, \quad k \in \mathcal{E}_d \setminus \{0\}, \quad 1 \leq i \leq h_{m,k}.$$

By Proposition 4.1.5 the required cubature formula is valid since the polynomials $S_{ki}(x)$ constitute a basis in $\mathtt{Harm}(E;k)$. $\square$

COROLLARY 4.2.2. *Any spherical cubature formula of index $d = 2t$ is equivalent to the system of equalities*

$$\boxed{\sum_{x,y}\langle x, y\rangle^{2k}\varrho(x)\varrho(y) = \frac{1}{\Upsilon_{\mathbf{R}}(m,k)}, \quad 0 \leq k \leq t}\ . \tag{4.44}$$

where the constant $\Upsilon_{\mathbf{R}}(m,k)$ is described in Corollary 2.2.5.

Proof. The system (4.44) is equivalent to (4.41) being the restriction of (4.41) to the basis of $\Pi_d^{(\varepsilon_d)}$ consisting of the polynomials $F_k(u) = u^{2k}$, $0 \leq k \leq t$. Indeed,

$$c_{m,0}(F_k) = \int_{-1}^{1} u^{2k}\Omega_m(u)\,du = \frac{2}{\tau_m}\int_0^1 u^{2k}(1-u^2)^{\frac{m-3}{2}}\,du = \frac{1}{\Upsilon_{\mathbf{R}}(m,k)},$$

see Section 2.1 and formula (2.44). □

In particular, we have

COROLLARY 4.2.3. *A spherical code X is a spherical design of index $d = 2t$ if and only if*

$$\boxed{\sum_{x,y\in X} \langle x, y\rangle^{2k} = \frac{n^2}{\Upsilon_{\mathbf{R}}(m,k)}, \quad 0 \le k \le t} \ , \tag{4.45}$$

where $n = |X|$.

Corollary 4.2.3 was proven in [21] using a technique of tensor products, see also [29] for Corollary 4.2.2.

Corollary 4.2.2 allows us to effectively check any given ϱ as a candidate for a spherical cubature formula. For example, *any orthonormal basis in $\mathbf{R}^2$ is the spherical design of index* 2 since (4.44) with $X = \{e_1, e_2\}$ and $\varrho(e_1) = \varrho(e_2) = \frac{1}{2}$ is valid for $t = 1$ because of (2.46). Now we show that there is no other spherical cubature formulas of index 2 with two nodes on the circle $\mathbf{S}^1 = \mathbf{S}(\mathbf{R}^2)$. Indeed, according to Corollary 4.2.2 any spherical cubature formula of index 2 with nodes x, y is equivalent to the system of equalities

$$\begin{cases} \varrho_1 + \varrho_2 &= 1 \\ \varrho_1^2 + \varrho_2^2 + 2\varrho_1\varrho_2 \cos^2\vartheta &= \frac{1}{2} \end{cases}$$

where $\vartheta = \arccos\langle x, y\rangle$, $\varrho_1 = \varrho(x) > 0$, $\varrho_2 = \varrho(y) > 0$. The system implies $4\varrho_1\varrho_2 \sin^2\vartheta = 1$ while $\varrho_1\varrho_2 \le \frac{1}{4}$ so, $4\varrho_1\varrho_2 \sin^2\vartheta \le 1$. Hence $\varrho_1 = \varrho_2 = \frac{1}{2}$ and $\sin\vartheta = 1$, i.e. $x_1 \perp x_2$.

THEOREM 4.2.4. *Let a spherical cubature formula of even index d be valid. Then*

$$\boxed{\sum_x \varrho^2(x) \le \frac{\int_{-1}^{1} F(u)\Omega_m(u)\,du}{F(1)}} \ . \tag{4.46}$$

for any real polynomial $F \in \Pi_d^{(0)}$ such that $F(1) > 0$ and $F|A(X) \ge 0$. The equality is attained if and only if $F|A(X) = 0$.

Proof. By Lemma 4.2.1

$$\int_{-1}^{1} F(u)\Omega_m(u)\,du = F(1)\sum_x \varrho^2(x) + \sum_{x\ne y} F(\langle x, y\rangle)\varrho(x)\varrho(y) \ge F(1)\sum_x \varrho^2(x), \tag{4.47}$$

which results in the bound (4.46).

The equality in (4.46) is attained if and only if the sum over $x \ne y$ in (4.47) equals zero. This is equivalent to $f|A(X) = 0$ since x and y run over suppϱ. □

COROLLARY 4.2.5. *Let a spherical cubature formula of even index d be valid. Then*

$$\boxed{n \geq \frac{F(1)}{\displaystyle\int_{-1}^{1} F(u)\Omega_m(u)\,du}} \tag{4.48}$$

for any real polynomial $F \in \Pi_d^{(0)}$ *such that* $F(1) > 0$ *and* $F|A(X) \geq 0$.

Proof. Combine (4.46) with the elementary inequality

$$\sum_x \varrho(x) \leq \sqrt{n}\sqrt{\sum_x \varrho^2(x)} \tag{4.49}$$

and take (4.17) into account. □

PROPOSITION 4.2.6. (cf. Corollary 4.1.19). *The bound (4.48) is attained if and only if*

(i) *the weights are equal;*

(ii) *the polynomial F annihilates the angle set* $A(X)$.

Proof follows from inequality (4.49) and the relevant part of Theorem 4.2.4. □

In order to extract the best profit from (4.48) we have to solve the linear programming problem

$$\begin{cases} F \in \Pi_d^{(0)}, \ F(u) \geq 0 \ (-1 \leq u \leq 1), \\ \displaystyle\int_{-1}^{1} F(u)\Omega_m(u)du = 1, \\ F(1) \to \max \end{cases} \tag{4.50}$$

for even $d = 2t$. Indeed, the linear fractional functional (4.48) attains its maximum for $F \in \Pi_d^{(0)}$, $F \geq 0$, if and only if F is proportional to a solution of (4.50).

Below we consider the modified problem, namely,

$$\begin{cases} \Psi \in \Pi_t, \quad \Psi(u) \geq 0 \quad (-1 \leq u \leq 1), \\ \displaystyle\int_{-1}^{1} \Psi(u)\Omega_{\alpha,\beta}(u)du = 1, \\ \Psi(1) \to \max, \end{cases} \tag{4.51}$$

where $\Omega_{\alpha,\beta}$ is the normalized Jacobi weight, see (2.2). The solution is essentially given by Theorem 7.71.3 from [54] but the self-contained exposition below is preferable for our purposes.

According to the Markov-Lukács Theorem (see [54, Theorem 1.21.1]) a polynomial $\Psi(u) \in \Pi_t$ is nonnegative on $[-1,1]$ if and only if

$$\Psi(u) = (1+u)^{\varepsilon} P^2(u) + (1-u^{2-\varepsilon}) Q^2(u), \tag{4.52}$$

where P and Q are real polynomials of degrees $\le \left[\frac{t}{2}\right]$ and $\le \left[\frac{t}{2}\right] + \varepsilon - 1$ respectively, $\varepsilon = \varepsilon_t$. Therefore $\Psi(1) = 2^{\varepsilon} P^2(1)$ so, the value $\Psi(1)$ is independent of Q. On the other hand, for any fixed P the maximum of the fractional linear functional

$$\frac{\Psi(1)}{\int_{-1}^{1} \Psi(u) \Omega_{\alpha,\beta}(u) du} \tag{4.53}$$

is attained at $Q = 0$. Hence, $Q = 0$ for the maximizer $\Psi_{\max}$. For this reason we can replace (4.52) by

$$\Psi(u) = (1+u)^{\varepsilon} P^2(u). \tag{4.54}$$

Then

$$\int_{-1}^{1} \Psi(u) \Omega_{\alpha,\beta}(u)\, du = \int_{-1}^{1} P^2(u) \Omega_{\alpha,\beta}^{(\varepsilon)}(u)\, du, \tag{4.55}$$

where

$$\Omega_{\alpha,\beta}^{(\varepsilon)}(u) = (1+u)^{\varepsilon} \Omega_{\alpha,\beta}(u) = \frac{1}{\tau_{\alpha,\beta}} \omega_{\alpha,\beta+\varepsilon}(u). \tag{4.56}$$

Since $P(1)$ is a linear functional in the space $\Pi_{\left[\frac{t}{2}\right]}$, there exists a unique real polynomial $R \in \Pi_{\left[\frac{t}{2}\right]}$ such that

$$P(1) = (P, R) \tag{4.57}$$

where the inner product corresponds to the weight $\Omega_{\alpha,\beta}^{(\varepsilon)}$. By (4.55)

$$\int_{-1}^{1} \Psi(u) \Omega_{\alpha,\beta}(u) du = (P, P),$$

while

$$\Psi(1) = 2^{\varepsilon} P^2(1) = 2^{\varepsilon} (P, R)^2.$$

The problem (4.51) takes a simple geometrical form:

$$\begin{cases} (P, P) = 1, \\ |(P, R)| \to \max \end{cases} \tag{4.58}$$

in the space of all real polynomials of degree $\le \left[\frac{t}{2}\right]$. The only solution of (4.58) is

$$\boxed{P_{\max} = \widehat{R} = \pm \frac{R}{\|R\|}}\;. \tag{4.59}$$

The corresponding maximal value is

$$\Psi_{\max}(1) = 2^{\varepsilon} P_{\max}^2(1) = 2^{\varepsilon} \widehat{R}^2(1) = \frac{2^{\varepsilon} R^2(1)}{(R, R)} = 2^{\varepsilon} R(1). \tag{4.60}$$

Let $(T_k)_0^{\left[\frac{t}{2}\right]}$ be an arbitrary orthonormal real basis in $\Pi_{\left[\frac{t}{2}\right]}$. Then

$$R(u) = \sum_{k=0}^{\left[\frac{t}{2}\right]} (T_k, R) T_k(u) = \sum_{k=0}^{\left[\frac{t}{2}\right]} T_k(1) T_k(u). \tag{4.61}$$

In particular,

$$R(1) = \sum_{k=0}^{\left[\frac{t}{2}\right]} (T_k(1))^2 \tag{4.62}$$

and (4.60) turns into

$$\Psi_{\max}(1) = 2^{\varepsilon} \sum_{k=0}^{\left[\frac{t}{2}\right]} (T_k(1))^2. \tag{4.63}$$

Of course, the mostly relevant basis is

$$T_k = \lambda_k P_k^{(\alpha,\beta+\varepsilon)}, \quad 0 \le k \le \left[\frac{t}{2}\right], \tag{4.64}$$

where

$$\lambda_k = \frac{1}{\|P_k^{(\alpha,\beta+\varepsilon)}\|} \tag{4.65}$$

and the norm relates to the weight $\Omega_{\alpha,\beta}^{(\varepsilon)}$. Passing to the weight $\omega_{\alpha,\beta+\varepsilon}$ by (4.56) we obtain

$$\lambda_k = \sqrt{\tau_{\alpha,\beta}} \cdot \frac{1}{\|P_k^{(\alpha,\beta+\varepsilon)}\|_{\omega_{\alpha,\beta+\varepsilon}}}. \tag{4.66}$$

Hence,

$$\Psi_{\max}(1) = 2^{\varepsilon} \tau_{\alpha,\beta} \sum_{k=0}^{\left[\frac{t}{2}\right]} \left(\frac{(P_k^{(\alpha,\beta+\varepsilon)}(1))}{\|P_k^{(\alpha,\beta+\varepsilon)}\|_{\omega_{\alpha,\beta+\varepsilon}}} \right)^2,$$

i.e.

$$\boxed{\Psi_{\max}(1) = \Lambda_t^{(\alpha,\beta)}} \tag{4.67}$$

where $\Lambda_t^{(\alpha,\beta)}$ is defined by (2.16). The final result in this way is (2.18) which actually is an explicit expression for $\Psi_{\max}(1)$.

In order to find a maximizer $\Psi_{\max}$ we have to combine (4.54), (4.59), (4.61), (4.64), (4.65) and (4.66). Eventually, the maximizer is

$$(1+u)^{\varepsilon} \left(\sum_{k=0}^{\left[\frac{t}{2}\right]} \frac{P_k^{(\alpha,\beta+\varepsilon)}(u) P_k^{(\alpha,\beta+\varepsilon)}(1)}{\|P_k^{(\alpha,\beta+\varepsilon)}\|^2_{\omega_{\alpha,\beta+\varepsilon}}} \right)^2$$

up to proportionality. By (2.12) and (2.14)

$$\boxed{\Psi_{\max}(u) = \text{const} \cdot (1+u)^{\varepsilon} \left(P_{\left[\frac{t}{2}\right]}^{(\alpha+1,\beta+\varepsilon)}(u) \right)^2} \quad . \tag{4.68}$$

The original problem (4.50) can be easily reduced to (4.51). To this end we represent $F \in \Pi_d^{(0)}$ as

$$F(u) = \Phi(u^2) \tag{4.69}$$

where $\Phi \in \Pi_t,\ t = \frac{d}{2}$, the polynomial Φ is unique. With τ_m defined in (2.11) we have

$$\begin{aligned}\int_{-1}^{1} F(u)\Omega_m(u)\,du &= \frac{1}{\tau_m}\int_{-1}^{1} \Phi(u^2)(1-u^2)^{\frac{m-3}{2}}\,du = \frac{2}{\tau_m}\int_{0}^{1} \Phi(u^2)(1-u^2)^{\frac{m-3}{2}}\,du \\ &= \frac{1}{\tau_m}\int_{0}^{1} \Phi(v)v^{-\frac{1}{2}}(1-v)^{\frac{m-3}{2}}\,dv.\end{aligned}$$

The latter integral turns into

$$\frac{1}{2^{\frac{m-2}{2}}\tau_m}\int_{-1}^{1} \Psi(w)(1-w)^{\frac{m-3}{2}}(1+w)^{-\frac{1}{2}}\,dw$$

by setting $w = 2v - 1$ and

$$\Psi(w) = \Phi\left(\frac{1+w}{2}\right). \tag{4.70}$$

Finally,

$$\int_{-1}^{1} F(u)\Omega_m(u)\,du = \int_{-1}^{1} \Psi(w)\Omega_{\frac{m-3}{2},-\frac{1}{2}}(w)\,dw. \tag{4.71}$$

(The weight on the right hand side of (4.71) is normalized since (4.71) is true for $F = \Psi = \mathbf{1}$.)

The given by (4.69) correspondence $F \mapsto \Phi$ is an isomorphism $\Pi_d^{(0)} \to \Pi_t$ such that $\Psi(1) = F(1)$. Hence, the problem (4.50) is equivalent to the problem (4.51) with

$$\alpha = \frac{m-3}{2}, \quad \beta = -\frac{1}{2}, \tag{4.72}$$

so that

$$F_{\max}(u) = \Psi_{\max}(2u^2 - 1). \tag{4.73}$$

Now (4.67) yields

$$F_{\max}(1) = \Lambda_t^{(\frac{m-3}{2},-\frac{1}{2})}. \tag{4.74}$$

According to (2.19) and (2.20)

$$F_{\max}(1) = \Lambda_{\mathbf{R}}(m,t) = \binom{m+t-1}{m-1}. \tag{4.75}$$

This result combining with Corollary 4.2.5 yields Theorem 4.1.14 again but now we have a valuable additional information about equality conditions, i.e. for the tight spherical cubature formulas.

THEOREM 4.2.7. *If a spherical cubature formula of index $d = 2t$ is tight then*

(i) *the weights are equal, i.e. the support X is a spherical design;*

(ii) *the Gegenbauer polynomial $C_t^{\frac{m}{2}}(u)$ annihilates the angle set $A(X)$.*

Conversely, let X be a spherical code such that

$$|X| = \Lambda_{\mathbf{R}}(m, t)$$

and let the angle set $A(X)$ be annihilated by $C_t^{\frac{m}{2}}(u)$. Then X is a tight spherical design of index $d = 2t$.

Proof. The statement follows from Proposition 4.2.6 and formulae (4.68), (4.72) and (4.73). Indeed, this yields

$$F_{\max}(u) = \left(u^{\varepsilon} P_{\left[\frac{t}{2}\right]}^{\left(\frac{m-1}{2}, -\frac{1}{2}+\varepsilon\right)}(2u^2 - 1)\right)^2 .$$

By formula (4.1.5) from [54]

$$u^{\varepsilon} P_{\left[\frac{t}{2}\right]}^{\left(\frac{m-1}{2}, -\frac{1}{2}+\varepsilon\right)}(2u^2 - 1) = \text{const} \cdot P_t^{\left(\frac{m-1}{2}, \frac{m-1}{2}\right)}(u) = \text{const} \cdot C_t^{\frac{m}{2}}(u)$$

by (2.6). Finally,

$$\boxed{F_{\max}(u) = \text{const} \cdot \left(C_t^{\frac{m}{2}}(u)\right)^2} . \tag{4.76}$$

□

COROLLARY 4.2.8. *For any tight spherical design X of index $2t$ the angle set consists of $s \le t$ values, i.e.*

$$\boxed{|A(X)| \le t} . \tag{4.77}$$

Proof. It follows from **(ii)** and (2.7). □

Consider the simplest situation $t = 1$, i.e. $d = 2$.

COROLLARY 4.2.9. *A spherical code X is a tight spherical design of index 2 if and only if X is an orthonormal basis in E.*

Proof. In this case the lower bound (4.27) takes the form $n \ge m$, so $n = m$ in the tight situation. By Theorem (4.2.7) the set $X = (x_1, \ldots, x_m)$ is a tight spherical design of index 2 if and only if $A(X) = \{0\}$, i.e. $\langle x_i, x_k \rangle = 0$ for $i \ne k$. Indeed, $C_1^{\frac{m}{2}}(u) = mu$, see (2.6). □

Obviously, $\widetilde{X} = X \cup \{-X\}$ is a tight spherical 3-design but X is not a spherical design of index 4, otherwise, $n \ge (m+1)m/2$ for $t = 2$, $m \ge 2$.

4.3 Projective codes, cubature formulas and designs

From now on E will be a m-dimensional right Euclidean space over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, or $\mathbf{H}$.

A **projective code** X is a spherical code such that the projectivization $X \to \mathbf{P}(E)$ is bijective or, in other words, the points from X are projectively distinct. It is convenient for any spherical code Y to treat its **projectivization** as any projective code $X \subset Y$ consisting of all projectively distinct points from Y.

For a projective code X its **angle set** is defined as

$$\boxed{a(X) = \{2|\langle x, y\rangle|^2 - 1 : \; x, y \in X, \quad x \neq y\} \subset [-1, 1)} \quad . \tag{4.78}$$

Let us emphasize that $1 \notin a(X)$. Indeed, if $|\langle x, y\rangle| = 1 = \|x\| \cdot \|y\|$ for some $x, y \in X$ then $y = x\lambda$ with $\lambda \in U(\mathbf{K})$ but this contradicts to the definition of projective code. The difference between definitions (4.78) and (4.1) is motivated by the difference between (3.150) and (3.99).

The symmetrization of any projective code X yields $\widetilde{X}$, an antipodal spherical code. However, the projective angle set $a(X)$ is different from $A(\widetilde{X})/\mathbf{Z}_2$.

THEOREM 4.3.1. *Let X be a projective code, $|X| = n$, $|a(X)| = s$. Then*

$$n \leq \begin{cases} \binom{m+2s-1}{m-1} & (\mathbf{K} = \mathbf{R}) \\ \binom{m+s-1}{m-1}^2 & (\mathbf{K} = \mathbf{C}) \\ \dfrac{1}{2m-1}\binom{2m+s-2}{2m-2} \cdot \binom{2m+s-1}{2m-2} & (\mathbf{K} = \mathbf{H}) \end{cases} \tag{4.79}$$

Proof. Consider the polynomial f, $\deg f = s$, such that $f|a(X) = 0$. Then $f(1) \neq 0$ since $1 \notin a(X)$. Now we apply Lemma 3.3.6 and get

$$n \leq \sum_{k=0}^{s} \hbar_{m,2k} \tag{4.80}$$

like (4.4) in the proof of Theorem 4.1.1. By (3.117)

$$\boxed{n \leq \dim \mathtt{Pol}_{\mathbf{K};2s}(E)} \tag{4.81}$$

(cf. (4.5)). It remains to refer to Corollary 3.3.5. □

DEFINITION 4.3.2. *A* **projective cubature formula of index** $2t$ *is an identity*

$$\int \phi \, d\varrho = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E; 2t), \tag{4.82}$$

where σ is the normalized Lebesgue measure on the unit sphere $\mathbf{S}(E)$ and ϱ is a finitely supported measure such that the set

$$\operatorname{supp}\varrho = \{x_k\}_1^n \subset \mathbf{S}(E)$$

is a projective code and

$$\int \phi \, d\varrho \equiv \int_{\operatorname{supp}\varrho} \phi \, d\varrho = \sum_{x \in \operatorname{supp}\varrho} \phi(x)\varrho(x) \equiv \sum_x \phi(x)\varrho(x). \tag{4.83}$$

The points $x_1, \dots, x_n$ are called the **nodes** *and its measures $\varrho_k = \varrho(x_k)$, $1 \le k \le n$, are called the* **weights**. *The set* $\operatorname{supp}\varrho$ *is called the* **support of the projective cubature formula.**

Note that $\operatorname{supp}\varrho$ is podal in the real case. Actually, a real projective cubature formula of index $2t$ is the same as a podal spherical cubature formula of index $2t$. On the other hand, any spherical cubature formula of index $2t$ on $\mathbf{S}(E)$ over $\mathbf{K}$ generates (by projectivization) a projective cubature formula of the same index. Indeed,

$$\mathtt{Pol}_{\mathbf{K}}(E; 2t) \subset \mathtt{Pol}(E; 2t) \equiv \mathtt{Pol}_{\mathbf{R}}(E_{\mathbf{R}}; 2t). \tag{4.84}$$

The identity (4.82) can be also rewritten as

$$\sum_{k=1}^n \phi(x_k)\varrho_k = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E; 2t), \tag{4.85}$$

where $\varrho_k > 0, \quad 1 \le k \le n$.

The support of a projective cubature formula of index $2t$ in the case of equal weights is called a **projective design of index** $2t$.

The projective cubature formulas are also called the **weighted projective design**.

PROPOSITION 4.3.3. *A projective code X is a projective design of index $2t$ if and only if*

$$\sum_{x \in X} \phi(gx) = \sum_{x \in X} \phi(x), \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E; 2t), \tag{4.86}$$

for all $g \in U(E)$.

Proof is like for Proposition 4.1.4. (Note that $U(E)$ acts on $\mathbf{S}(E)$ transitively.) □

PROPOSITION 4.3.4 . *A projective cubature formula of index $2t$ is equivalent to the system of equalities*

$$\int \phi \, d\varrho = \sum_x \phi(x)\varrho(x) = 0, \quad \phi \in \mathtt{Harm}_{\mathbf{K}}(E; 2k), \quad 1 \le k \le t, \tag{4.87}$$

and

$$\int d\varrho = \sum_x \varrho(x) = 1. \tag{4.88}$$

Thus, *ϱ is normalized automatically.*

Proof is like for Proposition 4.1.5 with even d. □

COROLLARY 4.3.5. *A projective code X is a projective design of index $2t$ if and only if*

$$\sum_x \phi(x) = 0, \quad \phi \in \mathtt{Harm}_{\mathbf{K}}(E; 2k), \quad 1 \le k \le t. \tag{4.89}$$

COROLLARY 4.3.6. *A projective cubature formula of index $2t$ is the projective cubature formula of every index $2k$, $0 \le k \le t$.*

DEFINITION 4.3.7. *A* **projective cubature formula of degree** $2t$ *(or* **strength** $2t$*) is a projective cubature formula of all indices $2t, 2t-2, \dots, 0$, i.e.*

$$\int \phi \, d\varrho = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E; 2k), \quad 0 \le k \le t. \tag{4.90}$$

By Corollary 3.2.8 we have

PROPOSITION 4.3.8. *A projective cubature formula of degree $2t$ is the same as a projective cubature formula of index $2t$.*

A projective design of index $2t$ is called a **projective $2t$-design**.

In the lower bounds problem for the projective case we will follow the linear programming approach only. A simpler method (see proof of Theorem 4.1.14) needs some modification for $\mathbf{K} = \mathbf{C}$ and faces some difficulties for $\mathbf{K} = \mathbf{H}$. Let us start with the projective counterpart of Lemma 4.2.1.

LEMMA 4.3.9. *Any projective cubature formula of index $2t$ is equivalent to the identity*

$$\sum_{x,y} f(2|\langle x, y\rangle|^2 - 1)\varrho(x)\varrho(y) = c^{\delta}_{m,0}(f) = \int_{-1}^{1} f(u)\Omega_{\alpha,\beta}(u)\, du, \quad f \in \Pi_t, \tag{4.91}$$

where $\Omega_{\alpha,\beta}(u)$ is the normalized weight (2.2) with α and β given by (3.128).

Proof is like for Lemma 4.2.1 but with (3.150) instead of (3.99). $\square$

COROLLARY 4.3.10. *Any projective cubature formula of index $2t$ is equivalent to the system of equalities*

$$\boxed{\sum_{x,y} |\langle x, y\rangle|^{2k} \varrho(x)\varrho(y) = \frac{1}{\Upsilon_{\mathbf{K}}(m,k)}, \quad 0 \le k \le t}\ . \tag{4.92}$$

Proof is like for Corollary 4.2.2 but with the basis

$$f_k(u) = \left(\frac{1+u}{2}\right)^k, \quad 0 \le k \le t,$$

in Π_t. Also note that

$$\int_{-1}^{1} \left(\frac{1+u}{2}\right)^k \Omega_{\alpha,\beta}(u)\, du = \frac{1}{\Upsilon_{\mathbf{K}}(m,k)}$$

by (2.41) and (2.42).$\square$

Corollary 4.3.10 for $\mathbf{K} = \mathbf{R}$ coincides with Corollary 4.2.2. Similarly, we have the following generalization of Corollary 4.2.3.

COROLLARY 4.3.11. *A projective code X is a projective design of index $2t$ if and only if*

$$\boxed{\sum_{x,y\in X} |\langle x,y\rangle|^{2k} = \frac{n^2}{\Upsilon_{\mathbf{K}}(m,k)}, \quad 0 \le k \le t}\ , \tag{4.93}$$

where $n = |X|$.

The projective counterpart of Theorem 4.2.4 is

THEOREM 4.3.12. *Let a projective cubature formula of index $2t$ be valid. Then*

$$\boxed{\sum_x \varrho^2(x) \le \frac{\displaystyle\int_{-1}^{1} f(u)\Omega_{\alpha,\beta}(u)\,du}{f(1)}}\ . \tag{4.94}$$

for any real polynomial $f \in \Pi_t$ such that $f(1) > 0$ and $f|a(X) \ge 0$. The equality is attained if and only if $f|a(X) = 0$.

Proof. It follows from Lemma 4.3.9 like Theorem 4.2.4 follows from Lemma 4.2.1. □

COROLLARY 4.3.13. *Let a projective cubature formula of index $2t$ be valid. Then*

$$\boxed{n \ge \frac{f(1)}{\displaystyle\int_{-1}^{1} f(u)\Omega_{\alpha,\beta}(u)\,du}} \tag{4.95}$$

for any real polynomial $f \in \Pi_t$ such that $f(1) > 0$ and $f|a(X) \ge 0$.

Proof is the same as for Corollary 4.2.5. □
Like Proposition 4.2.6 we have

PROPOSITION 4.3.14. *The bound (4.94) is attained if and only if*

(i) *the weights are equal;*

(ii) $f|a(X) = 0$.

In order to derive the lower bound for the number n of nodes in a projective cubature formula we insert into (4.95) the value (4.67) of the maximum in the problem (4.51) with the corresponding α, β. By (2.19)

$$\boxed{n \ge \Lambda_{\mathbf{K}}(m,t)}\ . \tag{4.96}$$

Now Corollary 3.3.5 yields

THEOREM 4.3.15. *Let a projective cubature formula of index* $2t$ *be valid. Then*

$$n \geq \begin{cases} \binom{m+t-1}{m-1} & (\mathbf{K}=\mathbf{R}) \\ \binom{m+[\frac{t}{2}]-1}{m-1} \cdot \binom{m+[\frac{t+1}{2}]-1}{m-1} & (\mathbf{K}=\mathbf{C}) \\ \frac{1}{2m-1}\binom{2m+[\frac{t}{2}]-2}{2m-2} \cdot \binom{2m+[\frac{t+1}{2}]-1}{2m-2} & (\mathbf{K}=\mathbf{H}) \end{cases} . \tag{4.97}$$

For $\mathbf{K}=\mathbf{R}$ this estimate coincides with (4.27) applying to the podal case.

REMARK 4.3.16. *According to (3.145) the inequality (4.96) with even* t *can be rewritten in dimensional terms*

$$\boxed{n \geq \dim \mathtt{Pol}_{\mathbf{K}}(E;t) \quad (\varepsilon_t = 0)} . \tag{4.98}$$

DEFINITION 4.3.17. *A projective cubature formula of index* $2t$ *is called* **tight** *if the equality is attained in (4.96) or, equivalently, in (4.97).*

THEOREM 4.3.18. *If a projective cubature formula of index* $2t$ *is tight then*

(i) *the weights are equal, i.e. its support* X *is a projective design;*

(ii) *with* $\varepsilon = \varepsilon_t$ *the polynomial*

$$(1+u)^{\varepsilon} P_{[\frac{t}{2}]}^{(\alpha+1,\beta+\varepsilon)}(u) \tag{4.99}$$

annihilates the angle set $a(X)$.

Conversely, let X *be a projective code such that*

$$|X| = \Lambda_{\mathbf{K}}(m,t)$$

and let the angle set $a(X)$ *be annihilated by the polynomial (4.99). Then* X *is a tight projective* $2t$*-design.*

Proof follows from Proposition 4.3.14 and formula (4.68). □

COROLLARY 4.3.19. *The support of a tight projective cubature formula of index* $2t$ *is a tight projective* $2t$*-design.*

COROLLARY 4.3.20. *For any tight projective* $2t$*-design* X *the angle set* $a(X)$ *consists of* $s \leq (t+\varepsilon)/2$ *values,*

$$\boxed{|a(X)| \leq \frac{t+\varepsilon}{2}} . \tag{4.100}$$

Proof. It follows from **(ii)** and (2.4) since $[t/2] + \varepsilon = (t + \varepsilon)/2$. $\square$

COROLLARY 4.3.21. *A projective code X is a tight projective design of index* 2 *if and only if X is an orthonormal basis in E.*

For $\mathbf{K} = \mathbf{R}$ this statement coincides with Corollary 4.2.9.

Proof. For any field $\mathbf{K}$ the lower bound (4.97) turns into $n \geq m$. In addition, for $t = 1$ we have $a(X) = \{-1\}$ since the polynomial in Theorem 4.3.18 is now $1 + u$. By definition of the angle set $a(X)$ we get $\langle x, y \rangle = 0$ for all $x, y \in X$ $(x \neq y)$. $\square$

Now we consider a more complicated example.

EXAMPLE 4.3.22. We apply Theorem 4.3.18 to obtain some tight projective cubature formulas of index $2t = 6$ (tight projective 6-designs) for $m = 2$. The elements of the corresponding angle set have to be the roots of the polynomial

$$(1 + u) P_1^{(\frac{\delta}{2}, \frac{\delta}{2})}(u) = \left(1 + \frac{\delta}{2}\right)(1 + u)u,$$

see (2.3). Therefore, $a(X) \subset \{-1, 0\}$, i.e., for $x, y \in X$ $(x \neq y)$, the only possible values of $|\langle x, y \rangle|$ are 0 and $\frac{1}{\sqrt{2}}$ For any real projective 6-design the number of nodes must be $n \geq 4$ according to (4.97) with $\mathbf{K} = \mathbf{R}$, $m = 2$, $t = 3$. On the other hand, the set $X_{\mathbf{R}} = \{x_1, x_2, x_3, x_4\}$, where

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad x_4 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

is a tight real projective 6-design.

Over $\mathbf{C}$ the bound (4.97) with $m = 2$, $t = 3$ yields $n \geq 6$. The set $X_{\mathbf{C}} = X_{\mathbf{R}} \cup \{x_5, x_6\}$ where

$$x_5 = \begin{bmatrix} \frac{\mathbf{i}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad x_6 = \begin{bmatrix} -\frac{\mathbf{i}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

is a tight complex projective 6-design.

Finally, $n \geq 10$ over $\mathbf{H}$ and $X_{\mathbf{H}} = X_{\mathbf{C}} \cup \{x_7, x_8, x_9, x_{10}\}$ where

$$x_7 = \begin{bmatrix} \frac{\mathbf{j}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad x_8 = \begin{bmatrix} -\frac{\mathbf{j}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad x_9 = \begin{bmatrix} \frac{\mathbf{k}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad x_{10} = \begin{bmatrix} -\frac{\mathbf{k}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

is a tight quaternionic projective 6-design.$\square$

4.4 Existence theorems and corresponding upper bounds

We still do not know anything about existence of spherical or projective cubature formulas with an arbitrary parameters m, t . Now we show how this problem can be solved in a very simple way. We start with an immediate consequence of the Completeness Theorem 3.3.7.

LEMMA 4.4.1. *A projective cubature formula of index $2t$ is provided by its validity for all elementary polynomial functions*

$$x \mapsto |\langle y, x\rangle|^{2t}, \quad x \in \mathbf{S}(E); \quad y \in \mathbf{S}(E). \tag{4.101}$$

Now we prove

THEOREM 4.4.2. *For any $t \in \mathbf{N}$, $t \geq 1$, there exists a projective cubature formula of index $2t$ with n nodes,*

$$n \leq \begin{cases} \binom{m+2t-1}{m-1} & (\mathbf{K} = \mathbf{R}) \\ \binom{m+t-1}{m-1}^2 & (\mathbf{K} = \mathbf{C}) \\ \frac{1}{2m-1}\binom{2m+t-2}{2m-2} \cdot \binom{2m+t-1}{2m-2} & (\mathbf{K} = \mathbf{H}) \end{cases} . \tag{4.102}$$

Proof. The Hilbert Identity (2.47) shows that the function

$$y \mapsto \int |\langle y, x\rangle|^{2t}\, d\sigma(x), \quad y \in \mathbf{S}(E), \tag{4.103}$$

belongs to the closed convex hull of the family of elementary polynomial functions

$$y \mapsto |\langle y, x\rangle|^{2t}, \quad y \in \mathbf{S}(E); \quad x \in \mathbf{S}(E). \tag{4.104}$$

By the well-known Caratheodory Theorem there exists a number ν,

$$1 \leq \nu \leq \dim \mathtt{Pol}_{\mathbf{K}}(E; 2t), \tag{4.105}$$

and a subset $\{x_k\}_1^{\nu+1} \in \mathbf{S}(E)$ such that the function (4.103) is a convex combination of functions (4.104):

$$\int |\langle y, x\rangle|^{2t}\, d\sigma(x) = \sum_{k=1}^{\nu+1} |\langle y, x_k\rangle|^{2t} \varrho_k\ , \quad y \in \mathbf{S}(E),$$

where

$$\varrho_k \geq 0, \quad \sum_{k=1}^{\nu+1} \varrho_k = 1.$$

Without loss of generality one can assume that

$$\int |\langle y, x\rangle|^{2t}\, d\sigma(x) = \sum_{k=1}^{n} |\langle y, x_k\rangle|^{2t} \varrho_k\ , \quad y \in \mathbf{S}(E), \tag{4.106}$$

where $n \leq \nu + 1$,

$$\varrho_k > 0, \quad \sum_{k=1}^{n} \varrho_k = 1, \tag{4.107}$$

and the points $x_1, \dots, x_n$ are projectively distinct. Identity (4.106) says that the projective cubature formula

$$\int \phi(x)\, d\sigma(x) = \sum_{k=1}^{n} \phi(x_k)\varrho_k \tag{4.108}$$

is valid for all elementary polynomial functions (4.101). By Lemma 4.4.1 formula (4.108) is a projective cubature formula of index $2t$ with $n \le \nu + 1$ nodes.

Suppose that $n = \nu + 1$. Then $\varrho_k > 0$ for $1 \le k \le \nu + 1$ but by (4.105) the functions

$$|\langle y, x_1\rangle|^{2t}, \dots, |\langle y, x_n\rangle|^{2t}$$

are linearly dependent:

$$\sum_{k=1}^{n} |\langle y, x_k\rangle|^{2t}\alpha_k = 0 \tag{4.109}$$

with some real α_k, $1 \le k \le n$. It follows from (4.106) and (4.109) that

$$\int |\langle y, x\rangle|^{2t}\, d\sigma(x) = \sum_{k=1}^{n} |\langle y, x_k\rangle|^{2t}(\varrho_k - \xi\alpha_k) \tag{4.110}$$

for any real ξ. With

$$\xi^{-1} = \max\left\{\frac{\alpha_k}{\varrho_k} :\ 1 \le k \le n\right\} > 0,$$

the number of nodes in (4.110) reduces to some $\widetilde{n} \le n - 1$, i.e. $\widetilde{n} \le \nu$. If now the nodes are $x_1, \dots, x_{\widetilde{n}}$ then

$$\int \phi(x)\, d\sigma(x) = \sum_{k=1}^{\widetilde{n}} \phi(x_k)\widetilde{\varrho}_k, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E; 2t), \tag{4.111}$$

by Lemma 4.4.1 again. The condition $\sum \widetilde{\varrho}_k = 1$ is satisfied automatically since $\phi = \mathbf{1}$ is admissible.

In any case the desired formula can be constructed with $n \le \nu$ nodes. By (4.105)

$$\boxed{n \le \dim \mathtt{Pol}_{\mathbf{K}}(E; 2t)}\ . \tag{4.112}$$

It remains to express the latter dimension by Corollary 3.3.5. $\square$

DEFINITION 4.4.3. *Given $m = \dim E$ and an integer $t \ge 1$, a projective cubature formula of index $2t$ on $\mathbf{S}(E)$ is called* **minimal** *if the number of nodes is the minimal possible. This number will be denoted by $N_{\mathbf{K}}(m, 2t)$.*

Obviously, *any tight projective cubature formula is minimal.* (The converse is not true.)

Combining Theorems 4.4.2 and 4.3.15 we obtain

THEOREM 4.4.4. *The following bounds hold:*

$$\boxed{\binom{m+t-1}{m-1} \le N_{\mathbf{R}}(m, 2t) \le \binom{m+2t-1}{m-1}}\ , \tag{4.113}$$

and

$$\boxed{\binom{m+[\frac{t}{2}]-1}{m-1}\cdot\binom{m+[\frac{t+1}{2}]-1}{m-1}\le N_{\mathbf{C}}(m,2t)\le\binom{m+t-1}{m-1}^2}\quad, \tag{4.114}$$

and

$$\boxed{\frac{\binom{2m+[\frac{t}{2}]-2}{2m-2}\cdot\binom{2m+[\frac{t+1}{2}]-1}{2m-2}}{2m-1}\le N_{\mathbf{H}}(m,2t)\le\frac{\binom{2m+t-2}{2m-2}\cdot\binom{2m+t-1}{2m-2}}{2m-1}}\quad. \tag{4.115}$$

By the way, these inequalities suggest a natural extension of $N_{\mathbf{K}}(2,2t)$ to $m=1$, namely,

$$\boxed{N_{\mathbf{K}}(1,2t)=1}\quad. \tag{4.116}$$

We accept this as a convenient agreement. In addition, we know that

$$\boxed{N_{\mathbf{K}}(m,2)=m}\quad, \tag{4.117}$$

see Corollary 4.3.21.

PROPOSITION 4.4.5. *The function $N_{\mathbf{K}}(m,2t)$ is non-decreasing with respect to m and t separately.*

Proof. The inequality

$$\boxed{N_{\mathbf{K}}(m,2t-2)\le N_{\mathbf{K}}(m,2t)} \tag{4.118}$$

follows from Corollary 4.3.6 immediately.

In order to prove the inequality

$$\boxed{N_{\mathbf{K}}(m-1,2t)\le N_{\mathbf{K}}(m,2t)} \tag{4.119}$$

it is sufficient to show that *the cubature formula (4.85) can be reconstructed into a cubature formula for $\phi\in\mathtt{Pol}_{\mathbf{K}}(E_1;2t)$ where E_1 is a $(m-1)$-dimensional subspace of E.* To this end we introduce the function $\phi_1(x)=\phi(x')$ where x' is the orthogonal projection of x on E_1. Then $\phi_1\in\mathtt{Pol}_{\mathbf{K}}(E;2t)$ since the projection is a linear operator. Hence,

$$\sum_{k=1}^{n}\phi(x'_k)\varrho_k=\sum_{k=1}^{n}\phi_1(x_k)\varrho_k=\int_{\mathbf{S}(\mathbf{K}^m)}\phi_1\,d\sigma.$$

The latter integral is a linear functional of ϕ, so that

$$\int_{\mathbf{S}(\mathbf{K}^m)} \phi_1 \, d\sigma = \int_{\mathbf{S}(\mathbf{K}^{m-1})} \overline{\psi}\phi \, d\widehat{\sigma}$$

where $\psi \in \mathtt{Pol}_{\mathbf{K}}(E_1; 2t)$ and $\widehat{\sigma}$ is the normed Lebesgue measure on $\mathbf{S}(\mathbf{K}^{m-1})$. This functional is $U(E_1)$-invariant. Therefore, $\psi = \text{const}$ and, finally, $\psi = 1$ since the measures σ and $\widehat{\sigma}$ are both normed. As a result,

$$\sum_{k=1}^{n} \phi(x'_k)\varrho_k = \int_{\mathbf{S}(\mathbf{K}^{m-1})} \phi \, d\widehat{\sigma},$$

which is the desired cubature formula up to projectivization of the system $(x'_k)_1^n$. $\square$

Now we develop an inductive (with respect to m) construction of projective cubature formulas. In particular, this allows us to improve the upper bounds in some cases. Let $y = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbf{S}(\mathbf{K}^m)$ where $z \in \mathbf{K} \equiv \mathbf{R}^\delta$, $w \in \mathbf{K}^{m-1} \equiv \mathbf{R}^{(m-1)\delta}$. Denote by $\mathbf{S}_+(\mathbf{K})$ the upper hemisphere in $\mathbf{R}^\delta$,

$$\mathbf{S}_+(\mathbf{K}) = \{z \in \mathbf{S}(\mathbf{K}) : \ s_1(z) \geq 0\},$$

where $s_1(z)$ is sign of the first real coordinate of z. Then $z = \xi\overline{\theta}$, where $\xi = |z|s_1(z)$. Thus, $|\xi| = |z|$, $\text{sign}\xi = s_1(z)$, $|\theta| = 1$, so that

$$-1 \leq \xi \leq 1, \quad \theta \in \mathbf{S}_+(\mathbf{K}). \tag{4.120}$$

Obviously,

$$\rho = \|w\| = \sqrt{1 - |z|^2} = \sqrt{1 - \xi^2}. \tag{4.121}$$

LEMMA 4.4.6. *Let $\phi(y) = \phi(z, w)$ be a continuous $U(\mathbf{K})$-invariant function on the sphere $\mathbf{S}(\mathbf{K}^m)$. Then in notation of Lemma 2.2.2 with $l = \delta$*

$$\int_{\mathbf{S}(\mathbf{K}^m)} \phi \, d\widetilde{\sigma} = \int_{\mathbf{S}(\mathbf{K}^{m-1})} d\widetilde{\sigma}_{\delta m-\delta-1}(\widehat{w}) \int_{\mathbf{S}_+(\mathbf{K})} d\widetilde{\sigma}_{\delta-1}(\theta) \int_{-1}^{1} \phi(\xi, \widehat{w}\sqrt{1-\xi^2}\theta)\pi_{\alpha,\beta}(\xi) \, d\xi \tag{4.122}$$

where

$$\pi_{\alpha,\beta}(\xi) = (1 - \xi^2)^\alpha |\xi|^{2\beta+1} \tag{4.123}$$

and, as usual,

$$\alpha = \frac{\delta m - \delta - 2}{2}, \quad \beta = \frac{\delta - 2}{2}. \tag{4.124}$$

Note that

$$\int_{\mathbf{S}_+(\mathbf{R})} \psi(\theta) d\widetilde{\sigma}_0(\theta) = \psi(1) \tag{4.125}$$

for any ψ, see Remark 2.2.3.

Proof. Since the function ϕ is $U(\mathbf{K})$-invariant, we have

$$\phi(y) = \phi(\xi\overline{\theta}, w) = \phi(\xi, w\theta) = \phi(\xi, \widehat{w}\|w\|\theta) = \phi(\xi, \widehat{w}\sqrt{1-\xi^2}\theta)$$

because of (4.121). For the same reason formula (2.29) turns into

$$d\widetilde{\sigma}(y) = (1-\xi^2)^{\alpha}|\xi|^{2\beta+1}d\xi d\widetilde{\sigma}_{\delta-1}(\theta)d\widetilde{\sigma}_{\delta m-\delta-1}(\widehat{w}).$$

□

With this lemma we can prove

THEOREM 4.4.7. *Each real projective cubature formula of index 2t with n nodes on* $\mathbf{S}(\mathbf{K}^{m-1}) \equiv \mathbf{S}(\mathbf{R}^{\delta m-\delta})$ *generates a* $\mathbf{K}$*-projective cubature formula of the same index 2t with N nodes on* $\mathbf{S}(\mathbf{K}^m)$ *where*

$$N = (t+1)N_{\mathbf{R}}\left(\delta, 2\left[\frac{t}{2}\right]\right)n. \tag{4.126}$$

Proof. We apply (4.122) to $\phi(y) = |\langle x, y\rangle|^{2t}$. Setting

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad y = \begin{bmatrix} z \\ w \end{bmatrix}, \tag{4.127}$$

where $u, z \in \mathbf{K}, \quad v, w \in \mathbf{K}^{m-1}$ we get

$$\phi(y) = \phi(z, w) = |\overline{u}z + \langle v, w\rangle|^{2t},$$

whence

$$\phi(\xi, \widehat{w}\sqrt{1-\xi^2}\theta) = |\overline{u}\xi + \sqrt{1-\xi^2}\chi(w)\theta|^{2t}$$

where $\chi(w) = \langle v, \widehat{w}\rangle$. Hence,

$$\phi(\xi, \widehat{w}\sqrt{1-\xi^2}\theta) = \Big(\xi^2|u|^2 + (1-\xi^2)|\chi(w)|^2 + \xi\sqrt{1-\xi^2}\,\kappa(\chi(w),\theta)\Big)^t$$

where

$$\kappa(\chi, \theta) = \chi\theta u + \overline{\chi\theta u}. \tag{4.128}$$

With

$$A(\chi, \xi) = \xi^2|u|^2 + (1-\xi^2)|\chi|^2, \quad B(\xi) = \xi\sqrt{1-\xi^2}, \tag{4.129}$$

we obtain

$$\begin{aligned} \phi(\xi, \widehat{w}\sqrt{1-\xi^2}\theta) &= \Big(A(\chi(w),\xi) + B(\xi)\kappa(\chi(w),\theta)\Big)^t \\ &= \sum_{q=0}^{t}\binom{t}{q}A^{t-q}(\chi(w),\xi)B^q(\xi)\kappa^q(\chi(w),\theta). \end{aligned} \tag{4.130}$$

Therefore,

$$\int_{\mathbf{S}_+(\mathbf{K})}\kappa^{2h}(\chi,\theta)\,d\widetilde{\sigma}_{\delta-1}(\theta)\int_{-1}^{1}\phi(\xi, \widehat{w}\sqrt{1-\xi^2}\theta)\pi_{\alpha,\beta}(\xi)\,d\xi = \tag{4.131}$$

$$\sum_{h=0}^{[\frac{t}{2}]}\binom{t}{2h}\kappa^{2h}(\chi(w),\theta)\int_{-1}^{1}A^{t-2h}(\chi(w),\xi)B^{2h}(\xi)\pi_{\alpha,\beta}(\xi)\,d\xi,$$

since

$$\int_{-1}^{1} A^{t-q}(\chi(w),\xi)B^q(\xi)\pi_{\alpha,\beta}(\xi)\,d\xi = 0 \quad (\varepsilon_q = 1). \tag{4.132}$$

Indeed, $\pi_{\alpha,\beta}(\xi)$ and $A(\chi(w),\xi)$ are even functions of ξ but $B^q(\xi) = \xi^q(1-\xi^2)^{\frac{q}{2}}$ is odd for odd q.

The product $A^{t-2h}(\chi(w),\xi)B^{2h}(\xi)$ in (4.131) is a polynomial of ξ of degree $\leq 2t$. For the weight $\pi_{\alpha,\beta}(\xi)$ one can apply the Gauss-Jacobi quadrature formula (see [54, Section 3.4]) with $t+1$ nodes ξ_i and corresponding coefficients μ_i, so that

$$\int_{-1}^{1} A^{t-2h}(\chi(w),\xi)B^{2h}(\xi)\pi_{\alpha,\beta}(\xi)\,d\xi = \sum_{i=1}^{t+1} A^{t-2h}(\chi(w),\xi_i)B^{2h}(\xi_i)\mu_i. \tag{4.133}$$

(Recall that the set $(\xi_i)_1^{t+1}$ coincides with the set of roots of $(t+1)$-th orthogonal polynomial for the given weight.)

Now we note that $\theta \mapsto \kappa^{2h}(\chi,\theta)$, $\theta \in \mathbf{S}(\mathbf{K})$, is a real polynomial function of degree $2h$. For

$$\nu = N_{\mathbf{R}}\left(\delta, 2\left[\frac{t}{2}\right]\right) \tag{4.134}$$

there exists a real projective cubature formula of index $2\left[\frac{t}{2}\right]$ with some nodes $\theta_1,\dots,\theta_\nu$ on $\mathbf{S}_+(\mathbf{K}) \equiv \mathbf{S}_+(\mathbf{R}^\delta)$ and some positive coefficients $\lambda_1,\dots,\lambda_\nu$. Hence,

$$2\int_{\mathbf{S}_+(\mathbf{K})} \kappa^{2h}(\chi,\theta)\,d\widetilde{\sigma}_{\delta-1}(\theta) = \sum_{j=1}^{\nu} \kappa^{2h}(\chi,\theta_j)\lambda_j. \tag{4.135}$$

(The latter sum in the case $\mathbf{K} = \mathbf{R}$ contains the only term $\kappa^{2h}(\chi,1)\lambda_1$.)

It follows from (4.131), (4.133) and (4.135) that

$$2\int_{\mathbf{S}_+(\mathbf{K})} \kappa^{2h}(\chi,\theta)\,d\widetilde{\sigma}_{\delta-1}(\theta)\int_{-1}^{1}\phi(\xi,\widehat{w}\sqrt{1-\xi^2}\theta)\pi_{\alpha,\beta}(\xi)\,d\xi = \tag{4.136}$$

$$\sum_{\substack{0\leq q\leq t\\ \varepsilon_q=0}}\left\{\sum_{j=1}^{\nu}\sum_{i=1}^{t+1}\binom{t}{q}\kappa^q(\chi(w),\theta_j)A^{t-q}(\chi(w),\xi_i)B^q(\xi_i)\mu_i\lambda_j\right\}. \tag{4.137}$$

In fact, the summation in (4.137) can be extended to odd $q \leq t$ since in this case

$$\sum_{i=1}^{t+1} A^{t-q}(\chi,\xi_i)B^q(\xi_i)\mu_i = 0.$$

Indeed, it is known that for any even weight on $[-1,1]$ the support of the corresponding cubature formula is symmetric with respect to the origin. (If the number of nodes is odd then one of them is 0.) Moreover, the coefficients for any pair of opposite nodes are equal. It remains to recall that the function $\xi \mapsto A^{t-q}(\chi,\xi)B^q(\xi)$ is odd for odd q. Thus,

$$\int_{\mathbf{S}_+(\mathbf{K})} \kappa^{2h}(\chi,\theta)\,d\widetilde{\sigma}_{\delta-1}(\theta)\int_{-1}^{1}\phi(\xi,\widehat{w}\sqrt{1-\xi^2}\theta)\pi_{\alpha,\beta}(\xi)\,d\xi =$$

$$\frac{1}{2}\sum_{q=0}^{t}\sum_{j=1}^{\nu}\sum_{i=1}^{t+1}\binom{t}{q}\kappa^q(\chi(w),\theta_j)A^{t-q}(\chi(w),\xi_i)B^q(\xi_i)\mu_i\lambda_j\ . \tag{4.138}$$

In (4.138) the q-th summand is a real polynomial function of $\widehat{w}$ of degree $\leq 2t - q \leq 2t$. By assumption, there exists a real projective cubature formula of index $2t$ with some nodes $\widehat{w}_1, \ldots, \widehat{w}_n$ on $\mathbf{S}(\mathbf{K}^{m-1})$ and with some positive coefficients $\epsilon_1, \ldots, \epsilon_n$. (Actually, the nodes can be chosen on $\mathbf{S}_+(\mathbf{K}^{m-1})$.) Hence,

$$\int_{\mathbf{S}(\mathbf{K}^{m-1})} \kappa^q(\chi(w), \theta_j) A^{t-q}(\chi(w), \xi_i)\, d\widetilde{\sigma}_{\delta m - \delta - 1}(\widehat{w}) = \sum_{k=1}^{n} \kappa^q(\chi(\widehat{w}_k), \theta_j) A^{t-q}(\chi(\widehat{w}_k), \xi_i)\epsilon_k.$$

As a result, according to (4.122),

$$\begin{aligned} \int_{\mathbf{S}(\mathbf{K}^{m-1})} \phi\, d\widetilde{\sigma} &= \frac{1}{2}\sum_{q=0}^{t}\sum_{i=1}^{t+1}\sum_{j=1}^{\nu}\sum_{k=1}^{n}\binom{t}{q}\kappa^q(\chi(\widehat{w}_k), \theta_j)A^{t-q}(\chi(\widehat{w}_k), \xi_i)B^q(\xi_i)\mu_i\lambda_j\epsilon_k \\ &= \frac{1}{2}\sum_{i=1}^{t+1}\sum_{j=1}^{\nu}\sum_{k=1}^{n}\phi(\xi_i, \widehat{w}_k\sqrt{1-\xi_i^2}\theta_j)\mu_i\lambda_j\epsilon_k \end{aligned}$$

by reading of (4.130) in the opposite direction. Setting

$$y_{ijk} = \begin{bmatrix} \xi_i \\ \\ \widehat{w}_k\sqrt{1-\xi_i^2}\theta_j \end{bmatrix} \tag{4.139}$$

and

$$\varrho_{ijk} = \frac{1}{2}\mathrm{Area}\mathbf{S}(\mathbf{R}^{(m-1)\delta})\mu_i\lambda_j\epsilon_k = \frac{\pi^{\frac{(m-1)\delta}{2}}}{\Gamma\left(\frac{(m-1)\delta}{2}\right)}\mu_i\lambda_j\epsilon_k \tag{4.140}$$

we conclude that

$$\int \phi\, d\sigma = \sum_{i=1}^{t+1}\sum_{j=1}^{\nu}\sum_{k=1}^{n}\phi(y_{ijk})\varrho_{ijk} \tag{4.141}$$

for the normalized Lebesgue measure σ on $\mathbf{S}(\mathbf{K}^m)$. Since we have (4.141) for an arbitrary elementary polynomial function of degree $2t$, the cubature formula of the same form (4.141) is valid by Lemma 4.4.1. The number of nodes in (4.141) coincides with (4.126) because of (4.134) $\square$.

Let us stress that the previous inductive proof is constructive.

The most important consequence of Theorem 4.4.7 is

THEOREM 4.4.8. *The inequality*

$$\boxed{N_{\mathbf{K}}(m, 2t) \leq (t+1)N_{\mathbf{R}}\left(\delta, 2\left[\frac{t}{2}\right]\right)N_{\mathbf{R}}((m-1)\delta, 2t)} \tag{4.142}$$

holds.

For $\mathbf{K} = \mathbf{R}$ Theorem 4.4.7 turns into

THEOREM 4.4.9. *Assume that for given m, t there exists a real projective (or, equivalently, spherical podal) cubature formula of index $2t$ with n nodes on $\mathbf{S}^{m-1}$. Then there exists a real projective cubature formula of the same index $2t$ with*

$$N = (t+1)n \tag{4.143}$$

nodes on $\mathbf{S}^m = \mathbf{S}(\mathbf{R}^{m+1})$.

Indeed, $N_{\mathbf{R}}\left(1, 2\left[\frac{t}{2}\right]\right) = 1$ by (4.116).

COROLLARY 4.4.10. *The inequality*

$$\boxed{N_{\mathbf{R}}(m, 2t) \leq (t+1) N_{\mathbf{R}}(m-1, 2t)} \tag{4.144}$$

holds.

In particular,

$$N_{\mathbf{R}}(2, 2t) \leq t+1.$$

Comparing this to the lower bound

$$N_{\mathbf{R}}(2, 2t) \geq t+1$$

following from (4.97) we obtain the exact value of $N_{\mathbf{R}}(2, 2t)$. Thus, we have

COROLLARY 4.4.11.

$$\boxed{N_{\mathbf{R}}(2, 2t) = t+1}\ . \tag{4.145}$$

Formulas (4.139) and (4.140) show that (4.145) is realized by the projective cubature formula on $\mathbf{S}(\mathbf{R}^2) \equiv \mathbf{S}^1$, i.e. the unit circle in the plane $\mathbf{R}^2$,

$$\int_{\mathbf{S}^1} \phi \, d\sigma = \sum_{i=1}^{t+1} \phi(y_i) \varrho_i \tag{4.146}$$

with the nodes

$$\begin{bmatrix} \xi_i \\ \sqrt{1-\xi_i^2} \end{bmatrix}, \quad 1 \leq i \leq t+1.$$

Here ξ_i are the nodes of the Gauss-Jacobi quadrature formula on $[-1, 1]$ corresponding to the weight

$$\pi_{-\frac{1}{2}, -\frac{1}{2}}(\xi) = \frac{1}{\sqrt{1-\xi^2}}, \quad -1 \leq \xi \leq 1. \tag{4.147}$$

This is just the Jacobi weight $\omega_{-\frac{1}{2}, -\frac{1}{2}}(\xi) = \omega_2(\xi)$ (see (2.9)) or, the same, the **Chebyshev weight**. The corresponding orthogonal polynomials are $C_k^0(\xi)$ (see (2.6)) which are proportional to the **Chebyshev polynomials**

$$T_k(\xi) = \cos(k(\arccos \xi)), \quad k \in \mathbf{N}.$$

Therefore, the nodes in the formula (4.146) are

$$\begin{bmatrix} \cos\frac{(2i-1)\pi}{2(t+1)} \\ \\ \sin\frac{(2i-1)\pi}{2(t+1)} \end{bmatrix}, \quad 1 \le i \le t+1. \tag{4.148}$$

These points are those of vertices of a regular $(2t+2)$-gone which lie in the upper half-plane. Also note that in (4.146)

$$d\sigma = \frac{1}{2\pi} d\vartheta$$

where ϑ is the angle coordinate on the unit circle, $0 \le \vartheta < 2\pi$. The angles corresponding to the nodes y_i are

$$\vartheta_i = \frac{(2i-1)\pi}{2(t+1)}, \quad 1 \le i \le t+1. \tag{4.149}$$

Since the measure σ is normalized we have

$$\sum_{i=1}^{t+1} \varrho_i = 1$$

in (4.146). In fact,

$$\varrho_i = \frac{1}{t+1}, \quad 1 \le i \le t+1,$$

since the quadrature formula (4.146) is tight, see Corollary 4.1.19. Thus, in this case we have a spherical design of index $2t$ on $\mathbf{S}(\mathbf{R}^2)$.

COROLLARY 4.4.12. *Assume that for given m, t there exists a real projective cubature formula of index $2t$ with n nodes on $\mathbf{S}^{m-1}$. Then there exists a real projective cubature formula of the same index $2t$ with*

$$N = (t+1)^{M-m} n \tag{4.150}$$

nodes on $\mathbf{S}^{M-1}$, $M \ge m$.

COROLLARY 4.4.13. *The inequality*

$$\boxed{N_{\mathbf{R}}(M, 2t) \le (t+1)^{M-m} N_{\mathbf{R}}(m, 2t), \quad M \ge m} \;. \tag{4.151}$$

holds.

In particular,

$$N_{\mathbf{R}}(m, 2t) \le (t+1)^{m-1}. \tag{4.152}$$

Comparing this to the upper bound from (4.113) we obtain

$$N_{\mathbf{R}}(m, 2t) \le \min\left\{ (t+1)^{m-1}, \binom{m+2t-1}{m-1} \right\}. \tag{4.153}$$

Eventually, we have

THEOREM 4.4.14.
$$N_{\mathbf{R}}(m,2t) \le \begin{cases} (t+1)^{m-1}, & m \le 4, \\ \dbinom{m+2t-1}{m-1}, & m \ge 5. \end{cases} \tag{4.154}$$

Proof. The right hand side of (4.154) coincides with the minimum in (4.153). The latter is shown below.

If $m = 2$ then
$$t+1 \le 2t+1 = \binom{2t+1}{1}.$$
If $m = 3$ then
$$(t+1)^2 \le (t+1)(2t+1) = \binom{2t+2}{2}.$$
If $m = 4$ then
$$(t+1)^3 \le \frac{(t+1)(2t+1)(2t+3)}{3} = \binom{2t+3}{3}.$$
However, if $m = 5$ then
$$(t+1)^4 \ge \frac{(t+1)(t+2)(2t+1)(2t+3)}{6} = \binom{2t+4}{4}.$$

By induction we assume that
$$(t+1)^{m-1} \ge \binom{m+2t-1}{m-1} \tag{4.155}$$
for some $m \ge 5$. Then
$$(t+1)^m \ge (t+1)\binom{m+2t-1}{m-1} > \frac{2t+m}{m}\binom{m+2t-1}{m-1} = \binom{m+2t}{m}.$$
□

In order to apply Theorem 4.4.8 to the complex case we note that the factor $N_{\mathbf{R}}\left(2, 2\left[\frac{t}{2}\right]\right)$ in (4.142) is known by (4.145). Thus, we have

THEOREM 4.4.15. *The identity*
$$\boxed{N_{\mathbf{C}}(m,2t) \le (t+1)(\left[\tfrac{t}{2}\right]+1)N_{\mathbf{R}}(2(m-1),2t)} \tag{4.156}$$
holds.

Moreover, Theorem 4.4.7 allows us to formulate more general and, actually, constructive

THEOREM 4.4.16. *Assume that for given m, t there exists a real projective cubature formula of index $2t$ with n nodes on $\mathbf{S}^{2m-3}$. Then there exists a complex projective cubature formula of the same index $2t$ with*

$$N = (t+1)\left(\left[\frac{t}{2}\right]+1\right)n \tag{4.157}$$

nodes on $\mathbf{S}(\mathbf{C}^m)$.

For the quaternionic case we need to consider the factor $N_{\mathbf{R}}\left(4, 2\left[\frac{t}{2}\right]\right)$ in (4.142). The exact values of that are mostly unknown at present. The exceptions are only

$$N_{\mathbf{R}}(4,2) = 4, \quad N_{\mathbf{R}}(4,4) = 11. \tag{4.158}$$

The first one is a particular case of (4.117) and the second one is due to Stroud [52]. By the way, according to (4.113) $N_{\mathbf{R}}(4,4) \geq 10$. Thus, *Stroud's cubature formula is minimal but not tight.*

Our Theorem 4.4.14 yields

$$N_{\mathbf{R}}\left(4, 2\left[\frac{t}{2}\right]\right) \leq \left(\left[\frac{t}{2}\right]+1\right)^3$$

for all t. According to (4.142), we have

THEOREM 4.4.17. *The inequality*

$$N_{\mathbf{H}}(m, 2t) \leq (t+1)\left(\left[\frac{t}{2}\right]+1\right)^3 N_{\mathbf{R}}(4(m-1), 2t) \tag{4.159}$$

holds.

For $t < 20$, $t \neq 12, 13$, the coefficient $(t+1)\left(\left[\frac{t}{2}\right]+1\right)^3$ in (4.159) can be improved. The point is that some upper bounds for $N_{\mathbf{R}}(4, 2k)$, $2 \leq k \leq 9$, follow from known results.

EXAMPLE 4.4.18. Liouville (1859) proved an identity which can be rewritten as

$$\sum_{k=1}^{12} \langle x, y_k\rangle^4 = \frac{3}{2}, \quad x \in \mathbf{S}(E), \tag{4.160}$$

where $(y_k)_1^{12} \subset \mathbf{S}(E)$, namely,

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad y_k = \begin{bmatrix} 1/2 \\ \pm 1/2 \\ \pm 1/2 \\ \pm 1/2 \end{bmatrix}, \quad 5 \leq k \leq 12.$$

On the other hand,

$$\int \langle x, y\rangle^4 \, d\sigma(y) = \frac{1}{\Upsilon_{\mathbf{R}}(4,2)} = \frac{1}{8} \tag{4.161}$$

by the Hilbert identity (2.47) and formula (2.38). Comparing (4.160) to (4.161) we obtain

$$\frac{1}{12}\sum_{k=1}^{12} \langle x, y_k\rangle^4 = \int \langle x, y\rangle^4 \, d\sigma(y). \tag{4.162}$$

By Lemma 4.4.1 this means that $(y_k)_1^{12}$ is a real projective design of index $2t = 4$. We see that $N_{\mathbf{R}}(4,4) \leq 12$, cf. (4.158). □

Some identities of Liouville's type corresponding to $m = 4$ and $2t = 6, 8$ were found by Kempner (1912) and Hurwitz (1908) respectively. All these identities yield some real projective cubature formulas like before. In particular, the Hurwitz identity results in a real projective cubature formula of index 8 with 72 nodes. Salihov (1975) considered the cases $m = 4$, $2t = 10, 18$. First, he constructed a real projective cubature formula of index 10 with 60 nodes. The latter is also a projective cubature formula of index 8 by Corollary 4.3.6. This is better than the above mentioned consequence of Hurwitz's identity. The second Salihov result combined with Corollary 4.3.6 covers the indices $2t = 12, 14, 16$. Eventually, we have the following table:

$2t =$	4	6	8	10	12	14	16	18
$N_{\mathbf{R}}(4, 2t) \le$	11	24	60	60	360	360	360	360

. (4.163)

Let us compare these bounds to the lower bounds following from (4.113):

$2t =$	4	6	8	10	12	14	16	18
$N_{\mathbf{R}}(4, 2t) \ge$	10	20	35	56	84	120	165	220

. (4.164)

We see that the bounds (4.163) are not tight.

By (4.142) and (4.163) we have

THEOREM 4.4.19. *For $1 \le t < 20$ the inequality*

$$N_{\mathbf{H}}(m, 2t) \le L(t) N_{\mathbf{R}}(4(m-1), 2t) \tag{4.165}$$

holds with $L(t)$ defined by the table

t	2	3	4	5	6	7	8	9	10	11	12
$L(t)$	12	16	55	66	168	192	540	600	660	720	4680

, (4.166)

t	13	14	15	16	17	18	19
$L(t)$	5040	5400	5760	6120	6480	6840	7200

. (4.167)

To compare this result to general bound (4.159) we note that

$$L(t) < (t+1)\left(\left[\frac{t}{2}\right] + 1\right)^3 \quad (t \ne 12, 13). \tag{4.168}$$

Thus, (4.165) *is better than (4.159) for $t < 20$, $t \ne 12, 13$.*

Similarly to Theorem 4.4.16 we have in the quaternionic case the following

THEOREM 4.4.20. *Assume that for given m, t there exists a real projective cubature formula of index $2t$ with n nodes on $\mathbf{S}^{4m-5}$. Then there exists a quaternionic projective cubature formula of the same index $2t$ with*

$$N = \begin{cases} L(t)n, & 2 \le t < 20, \quad t \ne 12, 13 \\ (t+1)\left(\left[\frac{t}{2}\right] + 1\right)^3 n, & t \ge 20 \text{ or } t = 12, 13 \end{cases} \tag{4.169}$$

nodes on $\mathbf{S}(\mathbf{H}^m)$.

Now we discuss the existence of real projective cubature formulas with $m = 3$ and $m > 4$ and with number of nodes less than (4.154) guarantees. For $m = 2, 4$ it was already done.

First of all, we complement (4.163) by a table of parameters $(2t, m, n)$ for some real projective cubature formulas which are known or follow from those by Corollary 4.3.6.

$2t$	4	4	6	6	6	6	6	6	6	8	8	10
m	3	23	3	6	7	8	16	23	24	3	24	24
n	6	276	11	63	113	120	2160	2300	98280	16	98280	98280

(4.170)

In the case $(4, 3, 6)$ the support is (up to projectivization) the set of vertices of the regular icosahedron on the sphere $\mathbf{S}(\mathbf{R}^3)$ [17]. The cases $(4, 23, 276)$, $(6, 8, 120)$, $(6, 23, 2300)$, $(8, 24, 98280)$ are known from [14]. The triple $(3, 6, 11)$ was found in [44].

THEOREM 4.4.21. *There exist some real projective cubature formulas corresponding to any triple $(2t, m, n)$ from the following tables:*

$2t = 6$:

m	5	9	10	11	17	18	24	25	26	27
n	96	480	1920	7680	8640	34560	9200	36800	147200	588800

, (4.171)

$2t = 8$:

m	5	25	26
n	300	491400	2457000

, (4.172)

$2t = 10$:

m	5	6	25	26	27	28
n	360	2160	589680	3538080	21228480	127370880

, (4.173)

$2t = 14$:

m	5
n	2880

, $2t = 16$:

m	5
n	3240

, $2t = 18$:

m	5
n	3600

. (4.174)

Proof. For example, the first column of the table corresponding to $2t = 6$ follows from the column of (4.163) with $2t = 6$ and formula (4.150) for $m = 4$, $M = 5$. The same reason for $M = 6$ provides $N = 384$ which is worse then $n = 63$ from (4.170). As soon as $M \geq 7$ the corresponding N's become greater than the upper bound (4.154). For this reason we neglect the situation $M \geq 7$. Similarly, we use the part of (4.170) regarding to $2t = 6$ in order to prove Theorem 4.4.21 in this case. The remaining tables in Theorem 4.4.21 are of the same origin. □

There is the gap $2t = 12$ in Theorem 4.4.21 since the triple $(12, 4, 360)$ looses competition with (4.154).

COROLLARY 4.4.22. *The inequalities*

$$N_{\mathbf{R}}(5, 6) \leq 96, \quad N_{\mathbf{R}}(6, 6) \leq 384, \quad \ldots, \quad N_{\mathbf{R}}(5, 18) \leq 3600.$$

hold.

The case of index $2t = 4$ is not included into Theorem 4.4.21 because of some special circumstances we explain now. The point is that there are two known [29], [32] series of real projective cubature formulas corresponding to the following parameters:

$$2t = 4, \quad n = \frac{m(m+2)}{2}, \quad m = 2^{2k}, \quad k \geq 1, \tag{4.175}$$

or

$$2t = 4, \quad n = \frac{3}{4}m^2 + 3, \quad m = 2k + 2, k \text{ is prime power.} \tag{4.176}$$

Then Corollary 4.4.12 yields

THEOREM 4.4.23. *There exist the real projective cubature formulas of index 4 with*

$$m = 2^{2k} + q, \quad n = 2^{2k} \cdot 3^q (2^{2k-1} + 1), \quad k \geq 1, \quad q \geq 0, \tag{4.177}$$

or with

$$m = 2k + 2 + q, \quad n = 3^{q+1}((k+1)^2 + 1), \quad k \text{ is prime power}, \quad q \geq 0. \tag{4.178}$$

Let us compare (4.177) with the upper bound

$$n \leq \binom{m+3}{4}$$

following from Theorem 4.4.14. For any fixed q the inequality

$$\frac{3^q(m-q)(m-q+2)}{2} < \binom{m+3}{4} \tag{4.179}$$

is valid if m is big enough. (For example, if $q = 1, 2$ then (4.179) is true for all $m \geq q$.)This means that the projective cubature formulas given by (4.177) with big k and fixed q are better than the formulas following from Theorem 4.4.2. The same conclusion is true for (4.178).

COROLLARY 4.4.24. *The inequalities*

$$N_{\mathbf{R}}(2^{2k} + q, 4) \leq 2^{2k} \cdot 3^q (2^{2k-1} + 1), \quad k \geq 1, \quad q \geq 0, \tag{4.180}$$

and

$$N_{\mathbf{R}}(2k + 2 + q, 4) \leq 3^{q+1}((k+1)^2 + 1), \quad k \text{ is prime power}, \quad q \geq 0. \tag{4.181}$$

hold.

Note that, there is the following table of known concrete real projective cubature formulas:

$2t = 4$: (4.182)

m	3	4	5	6	7	23
n	6	11	16	22	28	276

(See [14] for $(7, 28)$ and [52] for $(5, 16)$ and $(6, 22)$.) However, using (4.182) and Corollary 4.4.12 we only obtain the real projective cubature formulas of index 4 which are worse than previous one.

The complex projective cubature formulas corresponding to the triples $(2t, m, n)$ below come from [26] (except for $(4, 2, 4)$ from [29]) and Corollary 4.3.6.

(4.183)

$2t$	4	4	4	4	4	4	4	6	6	6	6	6	8	10
m	2	3	4	5	8	9	28	2	3	4	6	12	12	12
n	4	9	20	45	64	90	4060	6	21	40	126	32760	32760	32760

In addition, Theorem 4.4.16 and the above accumulated information on real projective cubature formulas yields the following

THEOREM 4.4.25. *There exist complex projective cubature formulas corresponding to the triples* $(2t, m, n)$ *from the following table*

$2t$	6	6	6	8	10	14
m	5	9	13	13	13	10
n	960	17280	73600	1474200	1769040	63685440

.

Other complex projective cubature formulas arising from all above mentioned real projective cubature formulas with $m \equiv 0 (\bmod\ 2)$ yield the number of nodes greater than the upper bound (4.114) or the corresponding value from (4.183).

COROLLARY 4.4.26. *The inequalities*

$$N_{\mathbf{C}}(5,6) \leq 960, \quad N_{\mathbf{C}}(9,6) \leq 17280, \quad \ldots, \quad N_{\mathbf{C}}(10,14) \leq 63685440$$

hold.

Using the series (4.177), (4.178) in frameworks of the same Theorem 4.4.16 we get some series of complex projective cubature formulas of index $2t = 4$. Namely, we have

THEOREM 4.4.27. *There exist the complex projective cubature formulas of index* 4 *with*

$$m = 2^{2k-1} + q + 1, \quad n = 2^{2k+1} \cdot 3^{2q+1}(2^{2k-1} + 1), \quad k \geq 1, \quad q \geq 0, \tag{4.184}$$

or with

$$m = k + q + 2, \quad n = 2 \cdot 3^{2q+2}((k+1)^2 + 1), \quad k \ \ is \ \ prime \ \ power, \quad q \geq 0. \tag{4.185}$$

COROLLARY 4.4.28. *The inequalities*

$$N_{\mathbf{C}}(2^{2k-1} + q + 1, 4) \leq 2^{2k+1} \cdot 3^{2q+1}(2^{2k-1} + 1), \quad q \geq 0, \quad k \geq 1, \tag{4.186}$$

and

$$N_{\mathbf{C}}(k + q + 2, 4) \leq 2 \cdot 3^{2q+2}((k+1)^2 + 1), \quad q \geq 0, \quad k \ \ is \ \ prime \ \ power. \tag{4.187}$$

hold.

The known (up to application of Corollary 4.3.6) quaternionic projective cubature formulas are the following (see [26]):

$2t$	4	4	4	4	6	6	6	6	8	10
m	2	3	4	5	2	3	4	5	3	3
n	10	63	36	165	10	63	180	165	315	315

. (4.188)

In addition, using Theorem 4.4.20 and the equality $N_{\mathbf{R}}(24, 10) = 98280$ we obtain

THEOREM 4.4.29. *There exists a quaternionic projective cubature formula of index* 10 *with* 6486480 *nodes on* $\mathbf{S}(\mathbf{H}^7)$.

COROLLARY 4.4.30. *The inequality*

$$N_{\mathbf{H}}(7, 10) \leq 6486480 \tag{4.189}$$

holds.

Also, like Theorem 4.4.27 we obtain

THEOREM 4.4.31. *There exist the quaternionic projective cubature formulas of index* 4 *with*

$$m = 2^{2k-2} + q + 1, \quad n = 2^{2k+2} \cdot 3^{4q+1} \cdot (2^{2k-1} + 1), \quad k \geq 1, \quad q \geq 0, \tag{4.190}$$

or with

$$m = 2k + q + 2, \quad n = 3^{q+1}((k+1)^2 + 1), \tag{4.191}$$

where

$$k \ \ is \ \ prime \ \ power, \quad q \geq 0, \quad 2k + q + 2 \equiv 0 \ (\text{mod} 4). \tag{4.192}$$

COROLLARY 4.4.32. *The inequalities*

$$N_{\mathbf{H}}(2^{2k-2} + q + 1, 4) \leq 2^{2k+2} \cdot 3^{4q+1} \cdot (2^{2k-1} + 1), \quad k \geq 1, \quad q \geq 0, \tag{4.193}$$

and

$$N_{\mathbf{H}}(2k + q + 2, 4) \leq 3^{q+1}((k+1)^2 + 1) \tag{4.194}$$

where

$$k \ \ is \ \ prime \ \ power, \quad q \geq 0, \quad 2k + q + 2 \equiv 0 \ (\text{mod} 4), \tag{4.195}$$

hold.

4.5 Invariant cubature formulas

Here we consider the cubature formulas which are invariant with respect to a group action.

Let G be a finite subgroup of the unitary group $U(E)$. There is a natural action of G on $\mathbf{S}(E)$,

$$x \mapsto gx, \quad x \in \mathbf{S}(E), \quad g \in G. \tag{4.196}$$

A spherical code is called G-**invariant** if $GX = X$ under action (4.196).

For every point $x \in \mathbf{S}(E)$ its orbit Gx is the minimal G-invariant spherical code containing x. A G-invariant spherical code X is called G-**homogeneous** if it is an orbit, i.e. the action is transitive.

For any spherical code V the **orbit** GV is the minimal G-invariant spherical code containing V. Obviously, it is the union of orbits of all $x \in V$,

$$GV = \bigcup_{x \in V} Gx. \tag{4.197}$$

Note that the orbits Gx and Gy with $x \neq y$ are either disjoint or coincide. Thus, GV can be represented as the union of pairwise disjoint orbits of some points from V.

For any finitely supported measure ϱ on $\mathbf{S}(E)$ and any $g \in U(E)$ we denote by $g\varrho$ the measure on $\mathbf{S}(E)$ such that

$$\operatorname{supp}(g\varrho) = g(\operatorname{supp}\varrho), \quad (g\varrho)(x) = \varrho(g^{-1}x), \quad x \in \operatorname{supp}\varrho. \tag{4.198}$$

This definition is equivalent to the identity

$$\int \lambda \, d(g\varrho) = \int (g\lambda) \, d\varrho \tag{4.199}$$

where λ is a function on X and

$$(g\lambda)(x) = \lambda(gx),$$

as usual. Obviously, any spherical cubature formula (4.6) generates a family of spherical cubature formulas of the same index d,

$$\int \phi \, d(g\varrho) = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}(E; d), \quad g \in U(E), \tag{4.200}$$

since the Lebesgue measure σ is unitary invariant as well as the functional space $\mathtt{Pol}(E; d)$.

A measure ϱ is called G**-invariant** if $g\varrho = \varrho$ for all $g \in G$, or, in other words, the set $\operatorname{supp}\varrho$ is G-invariant and $\varrho(gx) = \varrho(x)$ for all $x \in \operatorname{supp}\varrho$ and all $g \in G$. The latter means that the function $x \mapsto \varrho(x)$ is constant on any orbit Gx, $x \in \operatorname{supp}\varrho$.

DEFINITION 4.5.1. *A spherical cubature formula (4.6) is called* G**-invariant** *if the measure ϱ is G-invariant.*

Obviously, if a spherical cubature formula is G-invariant and its support is G-homogeneous then the support is a spherical design.

LEMMA 4.5.2. *For any spherical cubature formula with G-invariant support there exists a G-invariant spherical cubature formula with the same support.*

Proof. By assumption, for all $g \in G$ the measures $g\varrho$ have the same support X. The averaged measure

$$\widetilde{\varrho} = \operatorname*{Ave}_{g \in G} [g\varrho] \equiv \frac{1}{|G|} \sum_{g \in G} g\varrho \tag{4.201}$$

is G-invariant, $\operatorname{supp}\widetilde{\varrho} = X$ and

$$\int \phi \, d\widetilde{\varrho} = \int \phi \, d\sigma$$

by (4.200) and (4.201). □

COROLLARY 4.5.3. *If the support X of a spherical cubature formula is G-homogeneous then X is a G-invariant spherical design.*

In order to effectively construct some G-invariant cubature formulas we introduce the space $\mathtt{Harm}_G(E;d)$ of the spherical harmonics of degree d which are also G-invariant, i.e.

$$\phi(gx) = \phi(x), \quad g \in G, \quad x \in \mathbf{S}(E). \tag{4.202}$$

Similarly, we define the space $\mathcal{H}_G(E;d)$ of G-invariant harmonic forms of degree d. Recall that, by restriction to $\mathbf{S}(E)$, the spherical harmonics of degree d bijectively correspond to the harmonic forms of the same degree. Moreover, the G-invariant spherical harmonics are just the restrictions of the G-invariant harmonic forms.

PROPOSITION 4.5.4. *A G-invariant finitely supported measure ϱ on $\mathbf{S}(E)$ defines an (automatically G-invariant) spherical cubature formula of index d if and only if the system of equalities*

$$\int \phi \, d\varrho = \sum_x \phi(x)\varrho(x) = 0, \quad \phi \in \mathtt{Harm}_G(E;k), \quad k \in \mathcal{E}_d \setminus \{0\}, \tag{4.203}$$

holds and, in addition,

$$\int d\varrho = \sum_x \varrho(x) = 1 \tag{4.204}$$

in the case of even d.

Proof. For any spherical cubature formula the equalities (4.203) and (4.204) are automatically valid by Proposition 4.1.5 since $\mathtt{Harm}_G(E;k) \subset \mathtt{Harm}(E;k)$.

Conversely, let (4.203) and (4.204) be valid. Let $\phi \in \mathtt{Harm}(E;k)$, $k \in \mathcal{E}_d \setminus \{0\}$. Then the function

$$\psi(x) = \operatorname*{Ave}_{g\in G}\,[\phi(gx)]$$

belongs to $\mathtt{Harm}_G(E;k)$. By assumption, ψ satisfies (4.203). Hence,

$$\begin{aligned} 0 &= \sum_x \psi(x)\varrho(x) = \operatorname*{Ave}_{g\in G}\left[\sum_x \phi(gx)\varrho(x)\right] = \operatorname*{Ave}_{g\in G}\left[\sum_x \phi(x)\varrho(g^{-1}x)\right] \\ &= \operatorname*{Ave}_{g\in G}\left[\sum_x \phi(x)(g\varrho)(x)\right] = \sum_x \phi(x)\varrho(x). \end{aligned} \tag{4.205}$$

It remains to refer to Proposition 4.1.5. □

COROLLARY 4.5.5. *A G-invariant spherical code X is a G-invariant spherical design of index d if and only if the system of equalities*

$$\sum_x \phi(x) = 0, \quad \phi \in \mathtt{Harm}_G(E;k), \quad k \in \mathcal{E}_d \setminus \{0\}, \tag{4.206}$$

holds.

COROLLARY 4.5.6. *The G-orbit of a point $x_0 \in \mathbf{S}(E)$ is a spherical design of index d if and only if all G-invariant harmonic forms of degrees $k \in \mathcal{E}_d \setminus \{0\}$ vanish at the point x_0.*

Proof. In the G-homogeneous case the system (4.206) is reduced to $\phi(x_0) = 0,\ \ \phi \in \mathtt{Harm}_G(E;k),\ \ k \in \mathcal{E}_d \setminus \{0\}$. $\square$

COROLLARY 4.5.7. *Let X be a G-invariant spherical code. Let d be a positive integer such that there are no G-invariant harmonic forms of degrees k with $k \in \mathcal{E}_d \setminus \{0\}$, i.e.*

$$\mathcal{H}_G(E;k) = 0, \quad k \in \mathcal{E}_d \setminus \{0\}. \tag{4.207}$$

Then X is a spherical design of index d.

The simplest example of an invariant spherical cubature formula is given below.

EXAMPLE 4.5.8. The group $U(\mathbf{R}^2)$ is actually the orthogonal group $O(2)$. Each finite subgroup $G \subset O(2)$ is either cyclic or it is dihedral, i.e. the direct product of a cyclic subgroup of an odd order and $\mathbf{Z}_2 = \{-e, e\}$. On the other hand, for any $k \geq 1$ the basis in the space $\mathcal{H}(\mathbf{R}^1; k)$ is

$$\{\Re\mathrm{e}(z^k), \ \ \Im\mathrm{m}(z^k)\}, \quad z = x + y\mathbf{i} \in \mathbf{C} \equiv \mathbf{R}^2.$$

Thus,

$$\mathcal{H}(\mathbf{R}^2, k) = \{\Re\mathrm{e}(\mu z^k) : \ \mu \in \mathbf{C}\}. \tag{4.208}$$

For an integer $l \geq 2$ we consider the cyclic subgroup $\mathbf{Z}_l = \left(a^j\right)_0^{l-1} \subset O(2)$ where

$$az = \exp\left(\frac{2\pi\mathbf{i}}{l}\right) z, \quad z \in \mathbf{C}.$$

If a nonzero function from (4.208) is G-invariant then

$$\Re\mathrm{e}\left(\mu \exp\left(\frac{2k\pi\mathbf{i}}{l}\right) z^k\right) = \Re\mathrm{e}(\mu z^k), \quad z \in \mathbf{C}.$$

This is equivalent to $\exp\left(\frac{2k\pi\mathbf{i}}{l}\right) = 1$, i.e. $k \equiv 0 (\mathrm{mod}\ l)$. Thus, there are no G-invariant harmonic forms of degrees k, $1 \leq k \leq l-1$.

By Corollary 4.5.7 *any regular l-gone $X_l \subset \mathbf{R}^2$ is a $\mathbf{Z}_l$-invariant spherical design of index $l-1$.* If l is odd (even) then X_l is podal (antipodal). *In the case of even l the "half" X_l^+ of X_l is a podal spherical design of index $l-2$.* In both last cases the index turns out to be even.

Thus, *if d is even integer ≥ 2 then X_{d+2}^+ and X_{d+1} are both spherical designs of index d in $\mathbf{R}^2$ with $\frac{d}{2}+1$ and $d+1$ nodes respectively.*

Let us emphasize that X_{2t+2} coincides with the spherical design (4.148) regarding to the Gauss-Jacobi quadrature formula with Chebyshev weight.

Finally, the dihedral subgroup $\mathbf{Z}_l \times \mathbf{Z}_2$, $l \equiv 1(\mathrm{mod}\ 2)$, yields the antipodal spherical design $X_l \cup \{-X_l\}$ of index $d = l-1$ with $2l = 2d+2$ nodes. $\square$

Now we pass to a projective version of the previous theory. Since

$$g(x\gamma) = (gx)\gamma, \quad g \in U(E), \quad x \in E, \quad \gamma \in U(\mathbf{K}), \tag{4.209}$$

the action of $U(E)$ on the sphere $\mathbf{S}(E)$ commutes with the action of $U(\mathbf{K})$. Hence, $U(E)$ naturally acts on the projective space $\mathbf{P}(E)$.

Let G be a finite subgroup of $U(E)$. A projective code X is called **G-invariant** if GX is projectively equivalent to X. In this case the group G acts on X, since *there is a unique point $g \circ x \in X$ which is projectively equivalent to gx,*

$$g \circ x \in X, \quad g \circ x = (gx)\mu, \quad \mu \in U(\mathbf{K}), \tag{4.210}$$

where the scalar factor μ depends on g and x.

As usual, the action (4.210) can be transferred to all functions λ on X, namely,

$$(g \circ \lambda)(x) = \lambda(g \circ x). \tag{4.211}$$

In this setting it is important that

$$\phi(g \circ x) = \phi(gx), \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E), \quad x \in X, \tag{4.212}$$

because of $U(\mathbf{K})$-invariance of the polynomial functions. Thus,

$$g \circ (\phi|X) = (g\phi)|X. \tag{4.213}$$

For any finitely supported measure on $\mathbf{S}(E)$ we define $g \circ \varrho$ like (4.198), i.e.

$$\mathrm{supp}(g \circ \varrho) = g \circ (\mathrm{supp}\varrho), \quad (g \circ \varrho)(x) = \varrho(g^{-1} \circ x), \quad x \in \mathrm{supp}\varrho. \tag{4.214}$$

This definition is equivalent to the identity

$$\int \lambda \, d(g \circ \varrho) = \int (g \circ \lambda) \, d\varrho \tag{4.215}$$

like (4.199). In particular, it follows from (4.213) and (4.215)

$$\int \phi \, d(g \circ \varrho) = \int (g\phi) \, d\varrho, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E). \tag{4.216}$$

Any projective cubature formula (4.82) generates a family of projective cubature formula of the same index $2t$,

$$\int \phi \, d(g \circ \varrho) = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(E; 2t), \tag{4.217}$$

by (4.216) and the unitary invariance of σ.

A measure ϱ is called **G-invariant** if $g \circ \varrho = \varrho$ for all $g \in G$, i.e. $\mathrm{supp}\varrho$ is a G-invariant projective code and $\varrho(g \circ x) = \varrho(x)$ for all $x \in \mathrm{supp}\varrho$ and all $g \in G$.

DEFINITION 4.5.9. *A projective cubature formula (4.82) is called* **G-invariant** *if the measure ϱ is G-invariant.*

The below listed statements can be proved similarly to their spherical counterparts but now we use Proposition 4.3.4 and take (4.217) into account, if any.

LEMMA 4.5.10. *For any projective cubature formula with G-invariant support there exists a G-invariant projective cubature formula with the same support.*

COROLLARY 4.5.11. *If the support X of a projective cubature formula is G-homogeneous then X is a G-invariant projective design.*

Now we introduce the space $\mathtt{Harm}_{\mathbf{K};G}(E;2k)$ of those $U(\mathbf{K})$-invariant spherical harmonics of degree $2k$ which are also G-invariant,

$$\mathtt{Harm}_{\mathbf{K};G}(E;2k) = \{\phi \in \mathtt{Harm}_{\mathbf{K}}(E;2k) : \ g\phi = \phi\}.$$

Similarly, we define the space $\mathcal{H}_{\mathbf{K};G}(E;2k)$ of those $U(\mathbf{K})$-invariant harmonic forms of degree $2k$ which are also G-invariant. The restriction of this space to the sphere coincides with $\mathtt{Harm}_{\mathbf{K};G}(E;2k)$.

PROPOSITION 4.5.12. *Let X be a G-invariant projective code and let ϱ be a G-invariant measure such that* $\mathrm{supp}\varrho = X$. *Then ϱ defines an (automatically G-invariant) projective cubature formula of index $2t$ if and only if the system of equalities*

$$\int \phi \, d\varrho = \sum_x \phi(x)\varrho(x) = 0, \quad \phi \in \mathcal{H}_{\mathbf{K};G}(E;2k), \quad 1 \le k \le t, \tag{4.218}$$

holds and, in addition,

$$\int d\varrho = \sum_x \varrho(x) = 1. \tag{4.219}$$

COROLLARY 4.5.13. *A G-invariant projective code X is a G-invariant projective $2t$-design if and only if the system of equalities*

$$\sum_x \phi(x) = 0, \quad \phi \in \mathcal{H}_{\mathbf{K};G}(E;2k), \quad 1 \le k \le t, \tag{4.220}$$

holds.

COROLLARY 4.5.14. *The projectivization of the G-orbit of a point $x_0 \in \mathbf{S}(E)$ is a projective $2t$-design if and only if all forms from $\mathcal{H}_{\mathbf{K};G}(E;2k)$, $1 \le k \le t$, vanish at the point x_0.*

Note that *each point $x \in Gx_0$ has the same number of projectively equivalent points in its orbit* because of (4.209).

COROLLARY 4.5.15. *Let X be a G-invariant projective code. Let t be a positive integer such that*

$$\mathcal{H}_{\mathbf{K};G}(E;2k) = 0, \quad 1 \le k \le t. \tag{4.221}$$

Then X is a projective $2t$-design.

EXAMPLE 4.5.16. If d is an even integer ≥ 2 then X_{d+2}^{+} from Example 4.5.8 is a real projective design of index d with $\frac{d}{2}+1$ nodes on $\mathbf{S}(\mathbf{R}^2)$. $\square$

Below is the list of real projective cubature formulas from tables (4.163) and (4.170) which are known as invariant with respect to some groups.

$2t$	m	n	G
4	3	6	icosahedral group
6	4	24	Weyl-Coxeter group $W(F_4)$
6	6	63	Weyl-Coxeter group $W(E_6)$
6	8	120	Weyl-Coxeter group $W(E_8)$
10	4	60	Weyl-Coxeter group $W(I_4)$
10	24	98280	Conway group

(4.222)

In this table the only maximal value of index $2t$ is indicated for any m, n. In all cases from (4.222) the supports are G-homogeneous. Actually, Example 4.5.16 and the table (4.222) illustrate the role of orbits of finite unitary subgroups in construction of some projective cubature formulas. This **orbit method** can be described as follows.

Suppose that

$$\mathcal{H}_{\mathbf{K};G}(E;2) = 0$$

and let t be maximal such that

$$\mathcal{H}_{\mathbf{K};G}(E;2k) = 0, \quad 1 \leq k \leq t.$$

Then by Corollary 4.5.15 the projectivization of any G-orbit is a projective design of index $2t$.

Now one can try to find an orbit which yields a projective design of index $2(t+1)$. To this end one have to provide the system of equalities

$$\phi(x_0) = 0, \quad \phi \in \mathcal{H}_{\mathbf{K};G}(E;2t+2), \tag{4.223}$$

for the starting point x_0 of the orbit (Corollary 4.5.14). The system (4.223) can be reduced to a finite subsystem corresponding to all ϕ from a basis of $\mathcal{H}_{\mathbf{K};G}(E;2t+2)$.

All three 6-designs from Example 4.3.22 are projectivizations of some orbits. Respectively, the groups are: the group of regular 8-gone ($\mathbf{K} = \mathbf{R}$), the binary tetrahedral group ($\mathbf{K} = \mathbf{C}$) and the group of a regular quaternionic polygone ($\mathbf{K} = \mathbf{H}$).

In the next Section we systematically apply the orbit method to the complex projective cubature formulas on $\mathbf{S}(\mathbf{C}^2)$. In this setting one can only consider the finite subgroups of the group $SU(2)$.

LEMMA 4.5.17. *For any finite subgroup $G \subset U(m)$ there exists a subgroup*

$$\widetilde{G} \subset SU(m) = \{g \in U(m) : \det g = 1\}$$

such that for any $x \in \mathbf{S}(\mathbf{C}^m)$ the orbits Gx and $\widetilde{G}x$ are projectively equivalent. Moreover, if m is even then the subgroup $\widetilde{G}$ contains $g = -e$, so that $\widetilde{G} \supset \mathbf{Z_2}$.

Proof. Let $\gamma \in U(\mathbf{C})$ and let

$$(\sqrt[m]{\gamma})_k = \exp\left(\frac{\mathrm{Arg}\gamma}{m}\mathbf{i} + \frac{2\pi k}{m}\mathbf{i}\right)$$

where $0 \le \mathrm{Arg}\gamma < 2\pi$, $0 \le k \le m-1$. Consider

$$\widetilde{G} = \bigcup_{g\in G,\ 0\le k\le m-1} \frac{g}{(\sqrt[m]{\det g})_k}.$$

Obviously, $\widetilde{G}$ is a subset of $SU(m)$ and $G \subset \widetilde{G}$. Moreover, $\widetilde{G}$ is a subgroup of $SU(m)$. Indeed, for any $g, h \in G$ and $k, j \in \{0, 1, \ldots, m-1\}$ there exists $l \in \{0, 1, \ldots, m-1\}$ such that

$$\frac{g}{(\sqrt[m]{\det g})_k} \cdot \frac{h}{(\sqrt[m]{\det h})_j} = \frac{gh}{(\sqrt[m]{\det gh})_l}.$$

Further, for any $g \in G$ and $k \in \{0, 1, \ldots, m-1\}$ there exists $l \in \{0, 1, \ldots, m-1\}$ such that

$$\left(\frac{g}{(\sqrt[m]{\det g})_k}\right)^{-1} = g^{-1}(\sqrt[m]{\det g})_k = \frac{g^{-1}}{(\sqrt[m]{\det g^{-1}})_l}.$$

Obviously, for any $x \in \mathbf{S}(\mathbf{C}^m)$ and any $g \in G$ the points

$$\frac{gx}{(\sqrt[m]{\det g})_k}, \quad 0 \le k \le m-1,$$

are projectively equivalent to gx. This means that the orbits Gx and $\widetilde{G}x$ are projectively equivalent.

Finally, if m is even, then

$$\frac{e}{(\sqrt[m]{\det e})_{\frac{m}{2}}} = -e,$$

i.e. $-e \in \widetilde{G}$. $\square$

All finite subgroups of $SU(m)$ with $m = 1, 2$ are known. For $m = 1$ they are the cyclic subgroups of the unite circle. For $m = 2$ the complete list can be found in [50].

4.6 The orbit method based on subgroups of $SU(2)$.

`In this Section` $E = \mathbf{C}^2$ `and we use the notation`

$$x = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in \mathbf{C}^2 \tag{4.224}$$

`as a standard one.`

The following lemma allows us to construct the bases in the spaces $\mathcal{H}_{\mathbf{C};G}(E; 2k)$ we consider further. Note that

$$\dim \mathcal{H}_{\mathbf{C}}(E; 2k) = 2k + 1, \tag{4.225}$$

according to (3.143), (3.117) and (3.112).

LEMMA 4.6.1. *For $0 \le j \le k$ and $l, i \in \mathbf{N}$ let*

$$\alpha_{kj,l} = \frac{\prod_{i=0}^{l-1}(k-i)(j-i)}{\prod_{i=1}^{l} i(k+i-j)}, \quad \alpha_{kj,l+\frac{1}{2}} = 0, \tag{4.226}$$

$(\alpha_{kj,0=1})$ *and let*

$$\phi_{k,i,l}(x) = \xi_1^k \overline{\xi_2}^k |\xi_1|^{2i} |\xi_2|^{2l}. \tag{4.227}$$

Then $U(\mathbf{C})$-invariant forms

$$H_{kj} = \sum_{l=0}^{[\frac{j-1}{2}]} \alpha_{kj,l}\Big((-1)^l \phi_{k-j,j-l,l} + (-1)^{j-l}\phi_{k-j,l,j-l}\Big) + (-1)^{\frac{j}{2}} \alpha_{kj,\frac{j}{2}} \phi_{k-j,\frac{j}{2},\frac{j}{2}}, \tag{4.228}$$

together with

$$H_{k,j+k+1}(x) \equiv \overline{H_{kj}(x)} = H_{kj}(\overline{x}) \quad (0 \le j \le k-1) \tag{4.229}$$

constitute a basis of the space $\mathcal{H}_{\mathbf{C}}(E; 2k)$, $k \ge 1$.

Proof. By (4.228) and (4.227)

$$H_{kj}(x) = \sum_{l=0}^{j} (-1)^l \alpha_{kj,l} \xi_1^{k-l} \xi_2^l \overline{\xi_1}^{j-l} \overline{\xi_2}^{k+l-j}, \quad 0 \le j \le k. \tag{4.230}$$

According to (2.57),

$$\begin{aligned}
\frac{1}{4}\Delta H_{kj}(x) &= \frac{\partial^2 H_{kj}(x)}{\partial \xi_1 \overline{\xi_1}} + \frac{\partial^2 H_{kj}(x)}{\partial \xi_2 \overline{\xi_2}} \\
&= \sum_{l=0}^{j} (-1)^l \alpha_{kj,l}\Big((k-l)(j-l)\xi_1^{k-l-1}\xi_2^l \overline{\xi_1}^{j-l-1}\overline{\xi_2}^{k+l-j} \\
&+ l(k+l-j)\xi_1^{k-l}\xi_2^{l-1}\overline{\xi_1}^{j-l}\overline{\xi_2}^{k+l-j-1}\Big) \\
&= \sum_{l=0}^{j} (-1)^l \left(\lambda_{kj,l}\xi_1^{k-l-1}\xi_2^l \overline{\xi_1}^{j-l-1}\overline{\xi_2}^{k+l-j} + \mu_{kj,l}\xi_1^{k-l}\xi_2^{l-1}\overline{\xi_1}^{j-l}\overline{\xi_2}^{k+l-j-1}\right)
\end{aligned}$$

where

$$\lambda_{kj,l} = (k-l)(j-l)\alpha_{kj,l}, \quad \mu_{kj,l} = l(k+l-j)\alpha_{kj,l}. \tag{4.231}$$

Thus,

$$\frac{1}{4}\Delta H_{kj}(x) = \sum_{l=0}^{j-1} (-1)^l (\lambda_{kj,l} - \mu_{kj,l+1}) \xi_1^{k-l-1}\xi_2^l \overline{\xi_1}^{j-l-1}\overline{\xi_2}^{k+l-j}. \tag{4.232}$$

We show that

$$\lambda_{kj,l} - \mu_{kj,l+1} = 0, \quad 0 \le l \le j-1, \tag{4.233}$$

so, a fortiori, $\Delta H_{kj} = 0, \quad 0 \le j \le k$, and also $\Delta H_{k,j+k+1} = 0, \quad 0 \le j \le k-1$ by (4.229).

Obviously, $\mu_{kj,1} = \lambda_{kj,0}$ so, (4.233) is true for $l = 0$. For $1 \le l \le j-1$, the formulas (4.231) and (4.226) imply

$$\begin{aligned}\mu_{kj,l+1} &= (l+1)(k+l-j+1)\alpha_{kj,l+1} = (l+1)(k+l-j+1)\cdot\frac{\prod_{i=0}^{l}(k-i)(j-i)}{\prod_{i=1}^{l+1} i(k+i-j)} \\ &= \frac{\prod_{i=0}^{l}(k-i)(j-i)}{\prod_{i=1}^{l} i(k+i-j)} = (k-l)(j-l)\cdot\frac{\prod_{i=0}^{l-1}(k-i)(j-i)}{\prod_{i=1}^{l} i(k+i-j)} = \lambda_{kj,l}\ .\end{aligned}$$

Now note that all H_{kj} belong to the linear span of linearly independent monomials $\xi_1^{\alpha_1}\xi_2^{\alpha_2}\overline{\xi_1}^{\beta_1}\overline{\xi_2}^{\beta_2}$, $\alpha_1+\alpha_2=\beta_1+\beta_2=k$. Given $H_{kj}, \quad 0 \le j \le k$, in form (4.230), the monomial $\xi_1^k\overline{\xi_1}^j\overline{\xi_2}^{k-j}$ is contained in H_{kj} but not in $H_{ki}, \ 0 \le i \le 2k, \ i \ne j$. Thus, these forms are linearly independent. By (4.225) $(H_{kj})_{j=0}^{2k}$ is a basis of $\mathcal{H}_{\mathbf{C}}(E;2k)$. □

For any finite subgroup $G \subset SU(2)$ a basis of the spaces of all G-invariant harmonic forms of degree k can be obtained by G-averaging of $(H_{kj})_{j=0}^{2k}$.

`Later on` G `is a finite subgroup of` $SU(2)$ `containing` $g=-e$, see Lemma 4.5.17.

We denote by G^+ a subset of G consisting of all distinct representatives of $G/\mathbf{Z}_2$. Thus,

$$G = G^+ \cup G^-, \quad G^- = -G^+, \quad G^+ \cap G^- = \emptyset.$$

For definiteness we assume that $e \in G^+$. Then $-e \in G^-$.

PROPOSITION 4.6.2. *If $x \in \mathbf{S}(E)$ is not an eigenvector for any $g \in G$, $g \ne \pm e$, then the semiorbit G^+x is a G-invariant projective code.*

Proof. Let $g, h \in G$ and $gx = (hx)\gamma$, $\gamma \in U(\mathbf{C})$. Then $gx = h(x\gamma)$ and x turns to be an eigenvector for $h^{-1}g$. Hence, $h^{-1}g = \pm e$, or $g = \pm h$. Finally, $g = h$ since these elements are both from G^+. □

We start with $G = \mathcal{D}_2$ where $\mathcal{D}_2$ is the **binary dihedral group** consisting of 8 products $\pm a^k b^j$ $(k, j = 0, 1)$ where the generators a, b are

$$a = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \quad b = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \tag{4.234}$$

so that

$$a^2 = b^2 = -e$$

and

$$ab = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -ba.$$

In this situation one can take

$$\mathcal{D}_2^+ = \{e, a, b, ab\}.$$

In order to obtain a basis in the space $\mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;2k)$ of $\mathcal{D}_2$-invariant forms from $\mathcal{H}_{\mathbf{C}}(E;2k)$ we use the average

$$\operatorname*{Ave}_{\mathcal{D}_2}\,[\phi(x)] = \frac{1}{4}\Big(\phi(\xi_1,\xi_2) + \phi(\mathbf{i}\xi_1,-\mathbf{i}\xi_2) + \phi(\mathbf{i}\xi_2,\mathbf{i}\xi_1) + \phi(-\xi_2,\xi_1)\Big) \tag{4.235}$$

which is actually a projection $\mathcal{H}_{\mathbf{C}}(E;2k) \to \mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;2k)$. Thus, the system $\Big(\operatorname*{Ave}_{\mathcal{D}_2}\,[H_{kj}]\Big)_{j=0}^{2k}$ contains a basis of $\mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;2k)$.

Note that the expression (4.235) can be simplified by $U(\mathbf{C})$-invariance,

$$\operatorname*{Ave}_{\mathcal{D}_2}\,[\phi(x)] = \frac{1}{4}\Big(\phi(\xi_1,\xi_2) + \phi(\xi_1,-\xi_2) + \phi(\xi_2,\xi_1) + \phi(-\xi_2,\xi_1)\Big), \quad \phi \in \mathcal{P}_{\mathbf{C}}(E;2k). \tag{4.236}$$

LEMMA 4.6.3. *In notation of Lemma 4.6.1*

$$\operatorname*{Ave}_{\mathcal{D}_2}\,[\phi_{k,i,l}] = 0, \quad k \not\equiv 0 \pmod 2 \tag{4.237}$$

and

$$\operatorname*{Ave}_{\mathcal{D}_2}\,[\phi_{0,i,l-i} - \phi_{0,l-i,i}] = 0. \tag{4.238}$$

Proof follows from (4.227) and (4.236). □

LEMMA 4.6.4. *The forms*

$$I_1(x) = (\xi_1\overline{\xi_2})^2 + (\xi_2\overline{\xi_1})^2, \quad I_2(x) = |\xi_1|^4 + |\xi_2|^4 - 4|\xi_1\xi_2|^2 \tag{4.239}$$

constitute a basis in the space $\mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;4)$. The only basis form in $\mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;6)$ is

$$I_3(x) = (|\xi_1|^2 - |\xi_2|^2)\left((\xi_1\overline{\xi_2})^2 - (\xi_2\overline{\xi_1})^2\right). \tag{4.240}$$

In addition,

$$\mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;2) = 0. \tag{4.241}$$

Proof. By Lemma 4.6.1 and formula (4.237) we have

$$\operatorname*{Ave}_{\mathcal{D}_2}\,[H_{kj}] = 0, \quad 0 \le j \le k, \quad k \not\equiv j \pmod 2). \tag{4.242}$$

This yields (4.241). So, we have the only cases $k \equiv j \pmod 2$, i.e. $k = j$ or $k = 2$ and $j = 0$, or $k = 3$ and $j = 1$.

Let $k = j \equiv 1 \pmod 2$. Then (4.228) becomes

$$H_{jj} = \sum_{l=0}^{\frac{j-1}{2}} (-1)^l \alpha_{jj,l}\Big(\phi_{0,j-l,l} - \phi_{0,l,j-l}\Big).$$

Applying (4.238) we obtain

$$\operatorname*{Ave}_{\mathcal{D}_2}\,[H_{jj}] = 0, \quad j = 1, 3.$$

It remains to consider the averages of $H_{2,2},\ H_{2,0},\ H_{3,1}$ (see (4.229)). In such a way we obtain (4.239) and (4.240) up to proportionality. □

Now we apply Corollary 4.5.14 to obtain a complex projective 4-design. This design turns out to be tight.

THEOREM 4.6.5. *There exists a tight complex projective 4-design on* $\mathbf{S}(\mathbf{C}^2)$.

Proof. By Lemma 4.6.4 the common zeros on $\mathbf{S}(E)$ for $\mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;4)$ can be found from the system of equalities

$$\begin{cases} (\xi_1\overline{\xi_2})^2 + (\xi_2\overline{\xi_1})^2 &= 0 \\ |\xi_1|^4 + |\xi_2|^4 - 4|\xi_1\xi_2|^2 &= 0 \\ |\xi_1|^2 + |\xi_2|^2 &= 1. \end{cases}$$

The first equation is actually $I_1(x) = 0$, the second one is $I_2(x) = 0$.

Let $\xi_2 = \xi_1 z, \quad z = \rho e^{i\varphi}$. Then the system takes the form

$$\begin{cases} \overline{z}^2 + z^2 &= 0 \\ |z|^4 - 4|z|^2 + 1 &= 0 \\ |\xi_1|^2(1 + |z|^2) &= 1, \end{cases} \tag{4.243}$$

or

$$\begin{cases} \cos 2\varphi &= 0 \\ \rho^4 - 4\rho^2 + 1 &= 0 \\ |\xi_1|^2(1 + \rho^2) &= 1. \end{cases} \tag{4.244}$$

The values

$$\varphi = \frac{\pi}{4}, \quad \rho = \sqrt{2 + \sqrt{3}}, \quad \xi_1 = \frac{1}{\sqrt{3 + \sqrt{3}}}$$

satisfy (4.244). Respectively, with the same ρ, ξ_1 and

$$\boxed{\epsilon = \exp\left(\tfrac{\pi \mathbf{i}}{4}\right)} \tag{4.245}$$

the pair $z = \rho\epsilon,\ \xi_1$ satisfies (4.243).

By Corollary 4.5.14 the semiorbit

$$\mathcal{D}_2^+ x = \left\{ \begin{bmatrix} \xi_1 \\ \xi_1 z \end{bmatrix}, \quad \begin{bmatrix} \mathbf{i}\xi_1 \\ -\mathbf{i}\xi_1 z \end{bmatrix}, \quad \begin{bmatrix} \mathbf{i}\xi_1 z \\ \mathbf{i}\xi_1 \end{bmatrix}, \quad \begin{bmatrix} -\xi_1 z \\ \xi_1 \end{bmatrix} \right\}$$

is a projective 4-design. This is tight according to (4.97). □

`The notation (4.245) will be used throughout below.`

COROLLARY 4.6.6. *The equality*

$$\boxed{N_{\mathbf{C}}(2, 4) = 4} \tag{4.246}$$

holds.

Now let us consider the **binary tetrahedral group** $\mathcal{T} \subset SU(2)$, consisting of 24 products $\pm a^h b^j c^k$ $(h,\ j = 0,1;\ 0 \le k \le 2)$ where the generators a, b, c are

$$a = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \qquad c = \frac{1}{\sqrt{2}} \begin{bmatrix} \epsilon^7 & \epsilon^7 \\ \epsilon^5 & \epsilon \end{bmatrix} \tag{4.247}$$

(a and b are the same as in (4.234)). The following relations take place:

$$a^2 = b^2 = c^3 = -e, \quad ab = -ba, \quad ac = cb = bca.$$

Then

$$\mathcal{T}^+ = \{a^h b^j c^k :\ h,\ j = 0,1;\ 0 \le k \le 2\}. \tag{4.248}$$

LEMMA 4.6.7. *The only basis form in* $\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;6)$ *is*

$$I_3(x) = (|\xi_1|^2 - |\xi_2|^2)\left((\xi_1\overline{\xi_2})^2 - (\xi_2\overline{\xi_1})^2\right). \tag{4.249}$$

In addition,

$$\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;2) = 0, \quad \mathcal{H}_{\mathbf{C};\mathcal{T}}(E;4) = 0. \tag{4.250}$$

Proof. Since $\mathcal{D}_2 \subset \mathcal{T}$, all $\mathcal{T}$-invariant forms must be $\mathcal{D}_2$-invariant,

$$\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;2k) \subset \mathcal{H}_{\mathbf{C};\mathcal{D}_2}(E;2k).$$

For this reason $\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;2) = 0$ by (4.241).

In order to prove the second equality (4.250) we consider a linear combination

$$I = \lambda_1 I_1 + \lambda_2 I_2, \quad \lambda_1, \lambda_2 \in \mathbf{C}.$$

In particular, by (4.239)

$$I(\xi_1,\xi_1) = (\lambda_1 - 2\lambda_2)|\xi_1|^4, \quad I(\xi_1,0) = \lambda_2|\xi_1|^4.$$

On the other hand,

$$\begin{aligned} (cI)(\xi_1,\xi_2) \quad &= \quad -\frac{\lambda_1}{4}\left((\xi_1+\xi_2)^2(\overline{\xi_1}-\overline{\xi_2})^2 + (\xi_1-\xi_2)^2(\overline{\xi_1}+\overline{\xi_2})^2\right) \\ &+ \quad \lambda_2\Big(\frac{1}{4}|\xi_1+\xi_2|^4 + \frac{1}{4}|\xi_1-\xi_2|^4 - |(\xi_1+\xi_2)(\xi_1-\xi_2)|^2\Big), \end{aligned}$$

whence

$$(cI)(\xi_1,\xi_1) = 4\lambda_2|\xi_1|^4, \quad (cI)(\xi_1,0) = -\frac{1}{2}(\lambda_1+\lambda_2)|\xi_1|^4.$$

The system

$$\left\{ \begin{array}{rcl} (cI)(\xi_1,\xi_1) &=& I(\xi_1,\xi_1) \\ (cI)(\xi_1,0) &=& I(\xi_1,0) \end{array} \right.$$

is equivalent to

$$\left\{ \begin{array}{rcl} 4\lambda_2 &=& \lambda_1 - 2\lambda_2 \\ -\frac{1}{2}(\lambda_1+\lambda_2) &=& \lambda_2 \end{array} \right.$$

which implies $\lambda_1 = \lambda_2 = 0$. Thus, $\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;4) = 0$.

It remains to prove that I_3 is $\mathcal{T}$-invariant. It is sufficient to check that $cI_3 = I_3$ since already $aI_3 = I_3$ and $bI_3 = I_3$ by Lemma 4.6.4. We have,

$$\begin{aligned}
(cI_3)(\xi_1,\xi_2) &= I_3\left(\frac{\bar{\epsilon}}{\sqrt{2}}(\xi_1+\xi_2), \frac{\epsilon}{\sqrt{2}}(-\xi_1+\xi_2)\right) \\
&= -\frac{1}{8}\Big(|\xi_1+\xi_2|^2 - |\xi_1-\xi_2|^2\Big)\Big((\xi_1+\xi_2)^2(\overline{\xi_1}-\overline{\xi_2})^2 - (\xi_1-\xi_2)^2(\overline{\xi_1}+\overline{\xi_2})^2\Big) \\
&= -\frac{1}{8}\cdot 2(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2)\Big(\big((|\xi_1|^2-|\xi_2|^2)-(\xi_1\overline{\xi_2}-\xi_2\overline{\xi_1})\big)^2 - \big((|\xi_1|^2-|\xi_2|^2)+(\xi_1\overline{\xi_2}-\xi_2\overline{\xi_1})\big)^2\Big) \\
&= (\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2)(|\xi_1|^2-|\xi_2|^2)(\xi_1\overline{\xi_2}-\xi_2\overline{\xi_1}) = I_3(\xi_1,\xi_2).
\end{aligned}$$

□

THEOREM 4.6.8. *There exists a tight complex projective 6-design on* $\mathbf{S}(\mathbf{C^2})$.

Proof. The $\mathcal{T}$-invariant basis form I_3 vanishes at $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, an eigenvector of $a \in \mathcal{T}$. The projectivization of $\mathcal{T}^+x$ is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

By Corollary 4.5.14 this is a 6-design which is tight by (4.97). □

COROLLARY 4.6.9. *The equality*

$$\boxed{N_{\mathbf{C}}(2,6) = 6} \tag{4.251}$$

holds.

Now we pass to the **binary icosahedral group** $\mathcal{I} \subset SU(2)$. This group consists of 120 products $a^h, ba^h, a^hca^j, a^hcba^j (0 \le h \le 9,\ 0 \le j \le 4)$ where the generators a, b, c are

$$a = -\begin{bmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad c = \mu\begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix} \tag{4.252}$$

with

$$\eta = \exp\left(\frac{2\pi\mathbf{i}}{5}\right), \quad \mu = (\eta^2 - \eta^{-2})^{-1}, \quad \lambda = \eta + \eta^{-1}. \tag{4.253}$$

The basis relations are

$$aba = b, \quad bc = -cb, \quad cac = acba, \quad (a^2c)^2a^2 = c, \quad a^5 = b^2 = c^2 = -e.$$

Then

$$\mathcal{I}^+ = \{a^h, ba^h, a^hca^j, a^hcba^j :\ 0 \le h \le 4,\ 0 \le j \le 4\}.$$

Our first goal is to prove

PROPOSITION 4.6.10. $\mathcal{H}_{\mathbf{C};\mathcal{I}}(E;2k) = 0, \quad 1 \le k \le 5.$

To this end we need some auxiliary statements. We start with

LEMMA 4.6.11. *If* $\phi \in \mathcal{P}_{\mathbf{C}}(E;2k)$ *then*

$$\begin{aligned}\operatorname*{Ave}_{\mathcal{I}}\ [\phi] &= \frac{1}{60}\sum_{h=0}^{4}\Big(\phi(\eta^h\xi_1,\xi_2)+\phi(\xi_2,-\eta^h\xi_1) \\ &+ \sum_{j=0}^{4}\Big(\phi(\eta^h\mu(\eta^j\lambda\xi_1+\xi_2),\mu(\eta^j\xi_1-\lambda\xi_2))+\phi(\eta^h\mu(-\eta^j\xi_1+\lambda\xi_2),\mu(\eta^j\lambda\xi_1+\xi_2))\Big)\Big).\end{aligned} \tag{4.254}$$

Proof. For $0 \le h, j \le 4$ we set

$$\widetilde{a}_h = a^h = \eta^{2h}\begin{bmatrix}\eta^h & 0\\ 0 & 1\end{bmatrix}, \qquad \widetilde{b}_h = ba^h = \eta^{2h}\begin{bmatrix}0 & 1\\ -\eta^h & 0\end{bmatrix} \tag{4.255}$$

and

$$\widetilde{c}_{hj} = a^hca^j = \eta^{2h+2j}\mu\begin{bmatrix}\eta^{h+j}\lambda & \eta^h\\ \eta^j & -\lambda\end{bmatrix}, \quad \widetilde{d}_{hj} = a^hcba^j = \eta^{2h+2j}\mu\begin{bmatrix}-\eta^{h+j} & \eta^h\lambda\\ \eta^j\lambda & 1\end{bmatrix}. \tag{4.256}$$

Then

$$\operatorname*{Ave}_{\mathcal{I}}\ [\phi] = \operatorname*{Ave}_{\mathcal{I}^+}\ [\phi] = \frac{1}{60}\sum_{h=0}^{4}(\widetilde{a}_h\phi+\widetilde{b}_h\phi+\sum_{j=0}^{4}(\widetilde{c}_{hj}\phi+\widetilde{d}_{hj}\phi)),$$

or, equivalently,

$$\begin{aligned}\operatorname*{Ave}_{\mathcal{I}}\ [\phi] &= \frac{1}{60}\sum_{h=0}^{4}\Big(\phi(\eta^{3h}\xi_1,\eta^{2h}\xi_2)+\phi(\eta^{2h}\xi_2,-\eta^{3h}\xi_1)\\ &+ \sum_{j=0}^{4}\Big(\phi(\eta^{2h+2j}\mu(\eta^{j+h}\lambda\xi_1+\eta^h\xi_2),\eta^{2h+2j}\mu(\eta^j\xi_1-\lambda\xi_2))\\ &+ \phi(\eta^{2h+2j}\mu(-\eta^j\xi_1+\eta^h\lambda\xi_2),\eta^{2h+2j}\mu(\eta^j\lambda\xi_1+\xi_2))\Big)\Big).\end{aligned}$$

It remains to use $U(\mathbf{C})$-invariance of ϕ. $\square$

COROLLARY 4.6.12. *In notation of Lemma 4.6.1*

$$\operatorname*{Ave}_{\mathcal{I}}\ [\phi_{k,i,l}] = 0, \quad k \not\equiv 0 \pmod 5 \tag{4.257}$$

and

$$\operatorname*{Ave}_{\mathcal{I}}\ [\phi_{0,i,l-i} - \phi_{0,l-i,i}] = 0. \tag{4.258}$$

Proof. Applying (4.254) we see that $\operatorname*{Ave}_{\mathcal{I}}\,[\phi_{k,i,l}]$ is proportional to

$$\sum_{l=0}^{4}\eta^l = 0. \tag{4.259}$$

Let for short $\psi_{i,l} = \phi_{0,i,l-i} - \phi_{0,l-i,i}$. Then

$$\psi_{i,l}(\eta^l\xi_1, \xi_2) = -\psi_{i,l}(\xi_2, -\eta^l\xi_1) = -\psi_{i,l}(-\eta^l\xi_2, \xi_1).$$

This implies $\operatorname*{Ave}_{\mathcal{I}}\,[\psi_{i,l}] = 0$ by (4.254). □

In addition, we have

LEMMA 4.6.13. *Let $\mathcal{C}_5$ be a cyclic group $\{\widetilde{a}_h\}_0^4$. Then*

$$\operatorname*{Ave}_{\mathcal{C}_5}\,[\phi_{k,i,l}] = 0, \quad k \not\equiv 0 \pmod 5. \tag{4.260}$$

Proof follows from (4.255) and (4.259). □

The last lemma we need is quite elementary.

LEMMA 4.6.14.

$$|\mu|^4\lambda^2 = \frac{1}{5}, \tag{4.261}$$

$$|\mu|^4(\lambda^4 - 4\lambda^2 + 1) = -\frac{1}{5}, \tag{4.262}$$

$$|\mu|^8\left(\lambda^8 - 16\lambda^6 + 36\lambda^4 - 16\lambda^2 + 1\right) = -\frac{1}{5}. \tag{4.263}$$

$$|\mu|^{10}(1 + \lambda^{10}) = \frac{1}{5} \tag{4.264}$$

Proof. It is known that

$$\eta = \exp\left(\frac{2\pi\mathbf{i}}{5}\right) = \frac{\sqrt{5}-1}{4} + \mathbf{i}\sqrt{\frac{5+\sqrt{5}}{8}}. \tag{4.265}$$

Hence,

$$\lambda^2 = (\eta + \eta^{-1})^2 = 4\cos^2\frac{2\pi}{5} = \frac{(\sqrt{5}-1)^2}{4} = \frac{3-\sqrt{5}}{2}. \tag{4.266}$$

Thus,

$$|\mu|^4\lambda^2 = \frac{4\cos^2\frac{2\pi}{5}}{16\sin^4\frac{4\pi}{5}} = \frac{1}{64}(\sin\frac{2\pi}{5})^{-4}(\cos\frac{2\pi}{5})^{-2} = \frac{1}{64}\left(\frac{5+\sqrt{5}}{8}\cdot\frac{\sqrt{5}-1}{4}\right)^2 = \frac{1}{5},$$

i.e. (4.261) is true. Now

$$|\mu|^4 = \frac{1}{5\lambda^2} = \frac{2}{5(3-\sqrt{5})} = \frac{3+\sqrt{5}}{10} \tag{4.267}$$

and

$$\lambda^2+5|\mu|^4=\frac{3-\sqrt{5}}{2}+\frac{3+\sqrt{5}}{2}=3. \tag{4.268}$$

It follows from (4.267) and (4.266) that

$$|\mu|^2=\sqrt{\frac{3+\sqrt{5}}{10}}=\frac{5+\sqrt{5}}{10} \tag{4.269}$$

and

$$|\mu|^2(1+\lambda^2)=\frac{5+\sqrt{5}}{10}\left(1+\frac{3-\sqrt{5}}{2}\right)=1. \tag{4.270}$$

Using (4.261), (4.266) and(4.267) we obtain (4.262):

$$|\mu|^4(\lambda^4-4\lambda^2+1)=\frac{1}{5}\lambda^2-\frac{4}{5}+|\mu|^4=\frac{1}{5}\left(\frac{3-\sqrt{5}}{2}-4+\frac{3+\sqrt{5}}{2}\right)=-\frac{1}{5}.$$

Furthermore, by (4.262)

$$\begin{aligned}|\mu|^8\left(\lambda^8-16\lambda^6+36\lambda^4-16\lambda^2+1\right) &= |\mu|^8\Big((\lambda^4-4\lambda^2+1)^2-8\lambda^2+18\lambda^4-8\lambda^6)\Big)\\ &= \frac{1}{25}-2|\mu|^8\lambda^2(4-9\lambda^2+4\lambda^4).\end{aligned}$$

By (4.261) and (4.268)

$$\begin{aligned}&|\mu|^8\left(\lambda^8-16\lambda^6+36\lambda^4-16\lambda^2+1\right)=\\ &\frac{1}{25}-\frac{2}{5}|\mu|^4(4-9|\lambda|^2+4|\lambda|^4)=\frac{1}{25}-\frac{2}{5}\left(4|\mu|^4-\frac{9}{5}+\frac{4}{5}\lambda^2\right)=\\ &\frac{1}{25}-\frac{2}{25}\left(4(5|\mu|^4+\lambda^2)-9\right)=\frac{1}{25}-\frac{2}{25}(4\cdot 3-9)=-\frac{1}{5}.\end{aligned}$$

Finally,

$$|\mu|^{10}(1+\lambda^{10})=|\mu|^{10}(1+\lambda^2)(\lambda^8-\lambda^6+\lambda^4-\lambda^2+1)=|\mu|^{10}(\lambda^8-\lambda^6+\lambda^4-\lambda^2+1) \tag{4.271}$$

by (4.270). By (4.262) we get

$$|\mu|^8(\lambda^8+1)=\frac{1}{25}+|\mu|^8(8\lambda^6-18\lambda^4+8\lambda^2).$$

Substituting this in (4.271) and using (4.261) and (4.268) we obtain

$$|\mu|^{10}(1+\lambda^{10}) = \frac{1}{25}+|\mu|^8(7\lambda^6-17\lambda^4+7\lambda^2)=\frac{1}{25}+\frac{7}{25}(\lambda^2+5|\mu|^4)-\frac{17}{25}=-\frac{16}{25}+\frac{21}{25}=\frac{1}{5}.$$

□

Proof of Proposition 4.6.10. By Lemma 4.6.1 and (4.257) we have

$$\underset{\mathcal{I}}{\mathrm{Ave}}\,[H_{kj}]=0,\quad 0\le j\le k,\quad k\not\equiv j (\mathrm{mod}\ 5). \tag{4.272}$$

The only cases $k \equiv j (\text{mod } 5)$ are $k = j$ or $k = 5,\ j = 0$.

Let $k = j \equiv 1 (\text{mod } 2)$. Then (4.228) becomes

$$H_{jj} = \sum_{l=0}^{\frac{j-1}{2}} (-1)^l \alpha_{jj,l} \Big(\phi_{0,j-l,l} - \phi_{0,l,j-l} \Big).$$

Applying (4.258) we obtain

$$\operatorname*{Ave}_{\mathcal{I}} [H_{jj}] = 0, \quad j = 1, 3, 5.$$

It remains to consider the averages of $H_{2,2},\ H_{4,4},\ H_{5,0}$ (see (4.229)). The cyclic group $\mathcal{C}_5$ is a subgroup of $\mathcal{I}$. By Lemma 4.6.13 and Lemma 4.6.1 we conclude that $H_{2,2}$ is the only basis $\mathcal{C}_5$-invariant form from $\mathcal{H}_{\mathbf{C}}(E;4)$. But $H_{2,2}$ is not $\mathcal{I}$-invariant. Indeed, by (4.252) and (4.262)

$$(cH_{2,2})(1,0) = H_{2,2}(\mu\lambda, \mu) = |\mu|^4 (\lambda^4 - 4\lambda^2 + 1) = -\frac{1}{5}, \qquad H_{2,2}(1,0) = 1.$$

Hence, $\mathcal{H}_{\mathbf{C};\mathcal{I}}(E;4) = 0$.

Similarly, $H_{4,4}$ is the only basis $\mathcal{C}_5$-invariant form from $\mathcal{H}_{\mathbf{C}}(E;8)$. But $H_{4,4}$ is also not $\mathcal{I}$-invariant. Indeed, according to Lemma 4.6.1

$$H_{4,4}(\xi_1, \xi_2) = |\xi_1|^8 + |\xi_2|^8 - 16|\xi_1|^6|\xi_2|^2 - 16|\xi_1|^2|\xi_2|^6 + 36|\xi_1\xi_2|^4,$$

whence by (4.263)

$$(cH_{4,4})(1,0) = H_{4,4}(\mu\lambda, \mu) = |\mu|^8 \left(\lambda^8 - 16\lambda^6 + 36\lambda^4 - 16\lambda^2 + 1 \right) = -\frac{1}{5}, \quad H_{2,2}(1,0) = 1.$$

Hence, $\mathcal{H}_{\mathbf{C};\mathcal{I}}(E;8) = 0$.

Finally, we consider

$$H_{5,0}(x) = \left(\xi_1 \overline{\xi_2} \right)^5.$$

By (4.254)

$$\operatorname*{Ave}_{\mathcal{I}} [H_{5,0}] = \frac{1}{12} \Big(\xi_1^5 \overline{\xi_2}^5 + \xi_2^5 \overline{\xi_1}^5 + |\mu|^{10} f \Big), \tag{4.273}$$

where

$$f = \sum_{j=0}^{4} \Big((\eta^j \lambda \xi_1 + \xi_2)^5 (\overline{\eta}^j \overline{\xi_1} - \lambda \overline{\xi_2})^5 - (\eta^j \xi_1 - \lambda \xi_2)^5 (\overline{\eta}^j \lambda \overline{\xi_1} + \overline{\xi_2})^5 \Big).$$

It is easy to see that

$$f = \sum_{j=0}^{4} \Big((A + B_j)^5 - (A + \overline{B_j})^5 \Big)$$

where

$$A = \lambda(|\xi_1|^2 - |\xi_2|^2), \quad B_j = \overline{\eta^j} \xi_2 \overline{\xi_1} - \eta^j \lambda^2 \xi_1 \overline{\xi_2}.$$

Obviously,

$$f = \sum_{j=0}^{4} \sum_{r=0}^{5} \binom{5}{r} A^r \left(B_j^{5-r} - \overline{B_j}^{5-r} \right).$$

However,

$$\sum_{j=0}^{4}(B_j^k - \overline{B_j^k}) = 0, \quad 0 \le k \le 4,$$

by (4.259). Hence,

$$f = \sum_{j=0}^{4}(B_j^5 - \overline{B_j^5}) = 5(1+\lambda^{10})\left(\left(\xi_2\overline{\xi_1}\right)^5 - \left(\xi_1\overline{\xi_2}\right)^5\right).$$

By substitution in (4.273) we get

$$\underset{\mathcal{I}}{\text{Ave}}\, [H_{5,0}] = \frac{1}{12}\Big(1 - 5|\mu|^{10}(1+\lambda^{10})\Big)\Big(\left(\xi_1\overline{\xi_2}\right)^5 + \left(\xi_2\overline{\xi_1}\right)^5\Big) = 0$$

by (4.264). Thus, $\mathcal{H}_{\mathbf{C};\mathcal{I}}(E;10) = 0$. $\square$

Now we are in position to prove

THEOREM 4.6.15. *There exists a tight complex projective* 10*-design on* $\mathbf{S}(\mathbf{C}^2)$*.*

Proof. Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, an eigenvector of $a \in \mathcal{I}$. The projectivization of $\mathcal{I}^+x$ is

$$x_1 = x, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_j = \begin{bmatrix} \mu\lambda\eta^{j-3} \\ \mu \end{bmatrix}, \ 3 \le j \le 7; \quad x_k = \begin{bmatrix} -\mu\eta^{k-8} \\ \mu\lambda \end{bmatrix}, \ 8 \le k \le 12. \tag{4.274}$$

It remains to refer to Corollary 4.5.15 and to (4.97). $\square$

COROLLARY 4.6.16. *The equality*

$$\boxed{N_{\mathbf{C}}(2,10) = 12} \tag{4.275}$$

holds.

REMARK 4.6.17.Having (4.274) one can directly verify that this system is a complex projective 10-design on $\mathbf{S}(\mathbf{C}^2)$. To this end we apply Theorem 4.3.18 to the projective code $X = \{x_k\}_1^{12}$. First, we note that $|X| = \Lambda_{\mathbf{C}}(2,5)$, see (2.21). Further by (4.266) and (4.269) we have

$$|\langle x_1, x_j\rangle|^2 = |\langle x_2, x_{j+5}\rangle|^2 = |\mu^2|\lambda^2 = \frac{5-\sqrt{5}}{10}, \quad 3 \le j \le 7, \tag{4.276}$$

and

$$|\langle x_1, x_{j+5}\rangle|^2 = |\langle x_2, x_j\rangle|^2 = |\mu|^2 = \frac{5+\sqrt{5}}{10}, \quad 3 \le j \le 7. \tag{4.277}$$

For $3 \le j \le 7$ and $8 \le k \le 12$ we have

$$|\langle x_j, x_k\rangle|^2 = |\mu|^4|\lambda|^2|\eta^{k-j-5} - 1|^2 = \frac{1}{5}|\eta^{k-j-5} - 1|^2 = \begin{cases} 0 & k-j=5 \\ \frac{5-\sqrt{5}}{10} & k-j = 1,4,6,9 \\ \frac{5+\sqrt{5}}{10} & k-j=2,3,7,8 \end{cases} \tag{4.278}$$

by (4.265). Therefore, the corresponding angle set is

$$a(X) = \left\{ -\frac{\sqrt{5}}{5}, -1, \frac{\sqrt{5}}{5} \right\}. \tag{4.279}$$

Obviously, the polynomial

$$(1+u)P_2^{(1,1)}(u) = \frac{3}{4}(1+u)(5u^2-1) \tag{4.280}$$

annihilates $a(X)$. □

The Proposition 4.6.10 can not be extended to $k = 6$.

LEMMA 4.6.18. *The only basis form in* $\mathcal{H}_{\mathbf{C};\mathcal{I}}(E;12)$ *is*

$$I(x) = H_{6,6}(x) + 42\Big(H_{6,1}(x) + H_{6,1}(\overline{x})\Big). \tag{4.281}$$

Proof. By Lemma 4.6.1 and Corollary 4.6.12 the only averages of forms $H_{6,6}$, $H_{6,1}$ and $H_{6,7}(x) = H_{6,1}(\overline{x})$ can be different from zero. Averaging them by (4.254) we obtain (4.281) up to proportionality. (The corresponding calculations were done by computer.) □

Moreover, we have

LEMMA 4.6.19. $\mathcal{H}_{\mathbf{C};\mathcal{I}}(E;2k) = 0, \quad 7 \le k \le 9.$

Proof like before. □

THEOREM 4.6.20. *There exists a complex projective* 18*-design with* 60 *nodes on* $\mathbf{S}(\mathbf{C}^2)$.

Proof. By (4.281) some zeros of I can be found from the system of equations

$$\left\{ \begin{array}{rcl} H_{6,6}(x) & = & 0 \\ H_{6,1}(x) + H_{6,1}(\overline{x}) & = & 0. \end{array} \right. \tag{4.282}$$

According to Lemma 4.6.1 the system (4.282) can be rewritten as

$$\left\{ \begin{array}{r} |\xi_1|^{12} + |\xi_2|^{12} - 36\Big(|\xi_1|^{10}|\xi_2|^2 + |\xi_1|^2|\xi_2|^{10}\Big) + 225\Big(|\xi_1|^8|\xi_2|^4 + |\xi_1|^4|\xi_2|^8\Big) - 400|\xi_1\xi_2|^6 = 0 \\ \Big((\xi_1\overline{\xi_2})^5 + (\xi_2\overline{\xi_1})^5\Big)\Big(|\xi_1|^2 - |\xi_2|^2\Big) = 0 \end{array} \right. \tag{4.283}$$

The second equation (4.283) is satisfied by $\xi_1 = \mathbf{i}\sqrt{\rho}$, $\rho > 0$, $\xi_2 = 1$. In this way the first equation (4.283) becomes

$$\rho^6 - 36\rho^5 + 225\rho^4 - 400\rho^3 + 225\rho^2 - 36\rho + 1 = 0.$$

This equation has a positive solution ρ_0 since its left hand side is positive for $\rho = 0$ and negative for $\rho = 1$. As a result, $I(\mathbf{i}\sqrt{\rho_0}, 1) = 0$.

By Proposition 4.6.2 the $\mathcal{I}^+$-semiorbit of the vector

$$\widehat{x} = \begin{bmatrix} \mathbf{i}\sqrt{\frac{\rho_0}{1+\rho_0}} \\ \\ \frac{1}{\sqrt{1+\rho_0}} \end{bmatrix}$$

is a $\mathcal{I}$-invariant projective code consisting of 60 vectors since x is not an eigenvector of every $g \in \mathcal{I}^+$, $g \neq e$, see formulas (4.255), (4.256). By Lemmas 4.6.18, 4.6.19 and Corollary 4.5.15 the set $\mathcal{I}^+\widehat{x}$ is a design we need. $\square$

COROLLARY 4.6.21. *The inequality*

$$\boxed{30 \leq N_{\mathbf{C}}(2,18) \leq 60} \tag{4.284}$$

holds.

Proof by combination of Theorem 4.6.20 and lower bound (4.114). $\square$

Note that the upper bound (4.114) yields $N_{\mathbf{C}}(2,18) \leq 100$.

According to Corollary 4.3.6 we also have

COROLLARY 4.6.22. *The inequalities*

$$\boxed{20 \leq N_{\mathbf{C}}(2,14) \leq 60, \quad 25 \leq N_{\mathbf{C}}(2,16) \leq 60} \tag{4.285}$$

hold.

The upper bounds in (4.285) are also better than (4.114) yields.

In order to construct some further complex projective cubature formulas of index $2t$ we directly turn to the Proposition 4.5.12 instead of its corollaries.

Let $J_0, \ldots, J_{s-1}$ be the union of some bases of $\mathcal{H}_{\mathbf{C};G}(E; 2k)$, $1 \leq k \leq t$. We have to solve the system of equations

$$\sum_{i=1}^{\nu} J_j(x_i)\mu_i = 0, \quad 1 \leq j \leq s-1, \tag{4.286}$$

where $x_1, \ldots, x_\nu$ are the points on $\mathbf{S}(E)$ whose orbits form the support of a cubature formula, cf. (4.218). There are the additional constraints

$$\mu_i \geq 0, \quad \sum_{i=1}^{\nu} \mu_i > 0, \tag{4.287}$$

so that we have faced a linear programming problem. The necessary condition for existence of a solution is

$$\operatorname{rank} \begin{bmatrix} J_0(x_1) & \ldots & J_0(x_\nu) \\ J_1(x_1) & \ldots & J_1(x_\nu) \\ \ldots & \ldots & \ldots \\ J_{s-1}(x_1) & \ldots & J_{s-1}(x_\nu) \end{bmatrix} < \nu. \tag{4.288}$$

As a first example we consider the **binary dihedral group** $\mathcal{D}_4 \subset SU(2)$ consisting of 16 products $\pm a^k b^j$ $(0 \le k \le 3,\ j = 0,1)$ where the generators are

$$a = \begin{bmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon} \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}$$

and the relations are

$$a^4 = b^2 = -e, \quad aba = b.$$

Then

$$\mathcal{D}_4^+ = \{a^k b^j,\ 0 \le k \le 3,\ j = 0,1\}$$

and

$$\operatorname*{Ave}_{\mathcal{D}_4}\ [\phi] = \frac{1}{8}\sum_{k=0}^{3}\Big(\phi(\xi_1,\xi_2) + \phi(\xi_2,\xi_1)\Big). \tag{4.289}$$

LEMMA 4.6.23. *The only basis form in* $\mathcal{H}_{\mathbf{C};\mathcal{D}_4}(E;4)$ *is*

$$J_0(x) = |\xi_1|^4 + |\xi_2|^4 - 4|\xi_1\xi_2|^2. \tag{4.290}$$

The forms

$$J_1(x) = \frac{1}{2}\left((\xi_1\overline{\xi_2})^4 + (\xi_2\overline{\xi_1})^4\right) \tag{4.291}$$

and

$$J_2(x) = |\xi_1|^8 + |\xi_2|^8 - 16\left(|\xi_1|^6|\xi_2|^2 + |\xi_1|^2|\xi_2|^6\right) + 36|\xi_1\xi_2|^4 \tag{4.292}$$

constitute a basis in the space $\mathcal{H}_{\mathbf{C};\mathcal{D}_4}(E;8)$. *In addition,*

$$\mathcal{H}_{\mathbf{C};\mathcal{D}_4}(E;2) = 0. \tag{4.293}$$

Proof is standard taking into account that by (4.289) and (4.227)

$$\operatorname*{Ave}_{\mathcal{D}_4}\ [\phi_{k,i,l}] = 0, \quad k \not\equiv 0 \ (\mathrm{mod}\ 4) \tag{4.294}$$

and

$$\operatorname*{Ave}_{\mathcal{I}}\ [\phi_{0,i,l-i} - \phi_{0,l-i,i}] = 0. \tag{4.295}$$

□

THEOREM 4.6.24. *There exists a complex projective cubature formula of index* 8 *with* 10 *nodes on* $\mathbf{S}(\mathbf{C}^2)$.

Proof. Let $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, an eigenvector of $a \in \mathcal{D}_4$. For the vector $x_2 = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ the matrix

$$\begin{bmatrix} J_0(x_1) & J_0(x_2) \\ J_1(x_1) & J_1(x_2) \\ J_2(x_1) & J_2(x_2) \end{bmatrix} = \begin{bmatrix} 1 & J_0(x_2) \\ 0 & J_1(x_2) \\ 1 & J_2(x_2) \end{bmatrix} \tag{4.296}$$

must be of rank 1, i.e.

$$\begin{cases} J_0(x_2) &= J_2(x_2) \\ J_1(x_2) &= 0. \end{cases}$$

The first of these equations is

$$|\xi_1|^4 + |\xi_2|^4 - 4|\xi_1\xi_2|^2 = |\xi_1|^8 + |\xi_2|^8 - 16\left(|\xi_1|^6|\xi_2|^2 + |\xi_1|^2|\xi_2|^6\right) + 36|\xi_1\xi_2|^4.$$

By substitution $\rho = |\xi_1|^2$, $\xi_2 = 1$ we get

$$\rho(\rho^3 - 16\rho^2 + 35\rho - 16) = 0$$

which has a root $\rho_0 \in (2/5, 1)$. Then

$$x_2 = \begin{bmatrix} \rho_0 \\ \rho_0 e^{i\vartheta} \end{bmatrix}$$

and if $\vartheta = \frac{\pi \mathbf{i}}{8}$ then $J_1(x_2) = 0$ as well. Under this choice of x_2 the matrix (4.296) takes the form

$$\begin{bmatrix} 1 & J_0(x_2) \\ 0 & 0 \\ 1 & J_0(x_2) \end{bmatrix}.$$

Now the system (4.286) is equivalent to the single equation

$$\mu_1 + J_0(x_2)\mu_2 = 0.$$

Its solution

$$\mu_1 = -J_0(x_2), \quad \mu_2 = 1$$

is admissible. Indeed, $J_0(x_2) = \rho_0^2 - 4\rho_0 + 1 < 0$ and $\rho^2 - 4\rho + 1 < 0$ for $\rho \in (2-\sqrt{3}, 2+\sqrt{3}) \supset (2/5, 1)$.

Thus, by Proposition 4.5.12 the projectivization of the union of the semiorbits $\mathcal{D}_4^+ x_1$ and $\mathcal{D}_4^+ x_2$ is the support of a desired cubature formula. □

THEOREM 4.6.25. *The equality*

$$\boxed{N_{\mathbf{C}}(2, 8) = 10} . \tag{4.297}$$

holds.

In other words, *a complex projective cubature formula of index* 8 *with* 10 *nodes on* $\mathbf{S}(\mathbf{C}^2)$ *is minimal.*

Proof. It follows from Theorem 4.6.24 and (4.114) that

$$9 \le N_{\mathbf{C}}(2, 8) \le 10.$$

Assume that $N_{\mathbf{C}}(2, 8) = 9$. Then there exists a tight complex projective cubature formula with the triple of parameters $(2t, m, n) = (8, 2, 9)$. By Theorem 4.3.18 it is a projective design, say X, and the elements of the corresponding angle set $a(X)$ have to annihilate the polynomial

$$P_2^{(1,0)}(u) = \frac{1}{2}(5u^2 + 2u - 1),$$

see (2.3). Therefore, $a(X) \subset \{(-1+\sqrt{6})/5,\ (-1-\sqrt{6})/5\}$, i.e. for $x, y \in X$ ($x \neq y$) the only possible values of $|\langle x, y\rangle|^2$ are $(4+\sqrt{6})/10$ and $(4-\sqrt{6})/10$.

On the other hand,

$$\sum_{x,y\in X} |\langle x, y\rangle|^8 = \frac{81}{\Upsilon_{\mathbf{C}}(2,4)} = \frac{81}{5} \tag{4.298}$$

by Corollary 4.3.11 and formula (2.39). With the above information on the angle set formula (4.298) turns into

$$9 + \gamma_1 \left(\frac{4+\sqrt{6}}{10}\right)^4 + \gamma_2 \left(\frac{4-\sqrt{6}}{10}\right)^4 = \frac{81}{5} \tag{4.299}$$

where

$$\gamma_1,\ \gamma_2 \in \mathbf{N}, \quad \gamma_1 + \gamma_2 = 72.$$

This yields

$$\gamma_1 = \gamma_2 = \frac{9 \cdot 10^3}{217} \notin \mathbf{N},$$

the contradiction. □

Now let us come back to the tetrahedral group $\mathcal{T}$. In addition to Lemma 4.6.7 there is

LEMMA 4.6.26. *The only basis form in* $\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;8)$ *is*

$$I_4(x) = 16|\xi_1|^6|\xi_2|^2 + 16|\xi_1|^2|\xi_2|^6 - 36|\xi_1\xi_2|^4 - |\xi_1|^8 - |\xi_2|^8 - 5\Big((\xi_1\overline{\xi_2})^4 + (\xi_2\overline{\xi_1})^4\Big). \tag{4.300}$$

The forms

$$I_5(x) = (\xi_1\overline{\xi_2})^6 + (\xi_2\overline{\xi_1})^6 - \Big((\xi_1\overline{\xi_2})^2 + (\xi_2\overline{\xi_1})^2\Big)\Big(|\xi_1|^8 + |\xi_2|^8 - 8|\xi_1|^6|\xi_2|^2 - 8|\xi_1|^2|\xi_2|^6 + 15|\xi_1\xi_2|^4\Big) \tag{4.301}$$

and

$$\begin{aligned} I_6(x) \;&=\; 21\Big((\xi_1\overline{\xi_2})^4 + (\xi_2\overline{\xi_1})^4\Big)\Big(5|\xi_1|^4 + 5|\xi_2|^4 - 12|\xi_1\xi_2|^2\Big) \\ &-\; \Big(|\xi_1|^{12} - 36|\xi_1|^{10}|\xi_2|^2 + 225|\xi_1|^8|\xi_2|^4 - 400|\xi_1\xi_2|^6 + 225|\xi_1|^4|\xi_2|^8 - 36|\xi_1|^2|\xi_2|^{10} + |\xi_2|^{12}\Big) \end{aligned} \tag{4.302}$$

constitute a basis in the space $\mathcal{H}_{\mathbf{C};\mathcal{T}}(E;12)$.

Proof follows from Lemma 4.6.1 by averaging (partially with computer assistance). □

THEOREM 4.6.27. *There exists a complex projective cubature formula of index* 12 *with* 22 *nodes on* $\mathbf{S}(\mathbf{C}^2)$.

Proof. Let

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \sqrt{\frac{3-\sqrt{3}}{6}} \\ \sqrt{\frac{3+\sqrt{3}}{6}}\epsilon \end{bmatrix}, \quad x_3 = \begin{bmatrix} \rho \\ \sqrt{1-\rho^2}\epsilon \end{bmatrix}.$$

Here x_1 and x_2 are eigenvectors of b and $c \in \mathcal{T}$ respectively, see (4.247). The vectors x_1, x_2, x_3 annihilate I_5 and, in addition,

$$I_3(x_1) = I_5(x_1) = 0, \quad I_4(x_1) = I_6(x_1) = -1.$$

Hence, the matrix

$$w(r) = \begin{bmatrix} I_3(x_1) & I_3(x_2) & I_3(x_3) \\ I_4(x_1) & I_4(x_2) & I_4(x_3) \\ I_5(x_1) & I_5(x_2) & I_5(x_3) \\ I_6(x_1) & I_6(x_2) & I_6(x_3) \end{bmatrix} = \begin{bmatrix} 0 & I_3(x_2) & I_3(x_3) \\ -1 & I_4(x_2) & I_4(x_3) \\ 0 & 0 & 0 \\ -1 & I_6(x_2) & I_6(x_3) \end{bmatrix} \tag{4.303}$$

is of rank ≤ 2 if and only if

$$\begin{vmatrix} 0 & I_3(x_2) & I_3(x_3) \\ -1 & I_4(x_2) & I_4(x_3) \\ -1 & I_6(x_2) & I_6(x_3) \end{vmatrix} = 0.$$

This equation is algebraic of degree 8 with respect to ρ. One of its roots is $\rho_0 \approx 0.245275$. Below we take $\rho = \rho_0$ in x_3.

Now we have to solve the system of linear equations

$$\begin{cases} I_3(x_2)\mu_2 + I_3(x_3)\mu_3 &= 0 \\ -\mu_1 + I_4(x_2)\mu_2 + I_4(x_3)\mu_3 &= 0. \end{cases} \tag{4.304}$$

The system (4.304) has the positive solution, since

$$\frac{I_3(x_3)}{I_3(x_2)} = \sqrt{3}\rho_0^2(1-\rho_0^2)(2\rho_0^2 - 1) < 0$$

and

$$I_4(x_2) = \frac{187}{87}, \quad I_4(x_3) \approx 9.90703.$$

By Proposition 4.5.12 the union of the projectivizations of the semiorbits T^+x_1, T^+x_2 and T^+x_3 is the support of a desired cubature formula. Indeed, these projectivizations consist of 12, 6 and 4 points respectively. □

COROLLARY 4.6.28. *The inequality*

$$\boxed{16 \leq N_{\mathbf{C}}(2, 12) \leq 22} \tag{4.305}$$

holds.

4.7 Isometric embeddings $\ell_2^m \to \ell_p^n$

Consider an isometric embedding

$$f : \ell_2^m \to \ell_p^n, \quad n > m \geq 2, \quad p \neq 2,$$

over a classical field $\mathbf{K} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. *The number p must be even integer* otherwise such an embedding could not exist according to Theorem 1.10.4 .

We start with decomposition (1.86),

$$fx = \sum_{j=1}^{n} v_j \langle u_j, x \rangle,$$

corresponding to the canonical basis $(v_j)_1^n \subset \ell_p^n$. In our terminology the system $(u_j)_1^n$ is the frame of f. The isometry property

$$\|fx\|_p = \|x\|_2$$

in the coordinate form is equivalent to

$$\boxed{\sum_{j=1}^{n} |\langle u_j, x \rangle|^p = \langle x, x \rangle^{\frac{p}{2}}, \quad x \in \mathbf{K}^m} \quad . \tag{4.306}$$

This **basis identity** can be rewritten as

$$\boxed{\sum_{j=1}^{n} |\langle u_j, x \rangle|^p = \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right) \int |\langle y, x \rangle|^p \, d\sigma(y), \quad x \in \mathbf{K}^m,} \tag{4.307}$$

by the Hilbert identity (2.47). However, the vectors u_j in (4.307) can be not normalized. Moreover, some of them can be equal to zero so, the corresponding summands in (4.307) can be omitted. On the other hand, it is possible that there is a pair of proportional nonzero vectors among u_j's. In this case the corresponding summands can be brought together. For instance, if $u_2 = u_1\gamma$, $\gamma \in \mathbf{K}$, then

$$|\langle u_1, x \rangle|^p + |\langle u_2, x|^p = |\langle \widetilde{u}_1, x|^p \tag{4.308}$$

where

$$\widetilde{u}_1 = u_1(1 + |\gamma|^p)^{\frac{1}{p}}. \tag{4.309}$$

In the case of such a **reduction** of the embedding $\ell_2^m \to \ell_p^n$ the dimension n becomes less than the initial one. In the converse direction let us note that if there exists an isometric embedding $\ell_2^m \to \ell_p^n$ then its composition with the canonical isometric embedding $\ell_p^n \to \ell_p^N$, $N > n$, (see (1.114)) yields an isometric embedding $\ell_2^m \to \ell_p^N$.

If a system $(u_j)_1^n$ does not contain zeros and proportional pairs then it is called **irreducible**. Obviously, the normalization of an irreducible system yields a projective code on the unit sphere.

An isometric embedding $\ell_2^m \to \ell_p^n$ is called **irreducible** if its frame is irreducible.

There exists a close relation between isometric embeddings $\ell_2^m \to \ell_p^n$ and projective cubature formulas of index p.

THEOREM 4.7.1. *An isometric embedding $f : \ell_2^m \to \ell_p^n$ exists if and only if there exists a projective cubature formula of index p with some number $\nu \leq n$ of nodes on $\mathbf{S}(\mathbf{K}^m)$.*

Proof. Suppose that an isometric embedding $f : \ell_2^m \to \ell_p^n$ exists. After reduction we obtain an isometric embedding $\widetilde{f} : \ell_2^m \to \ell_p^\nu$, where $\nu \leq n$. For the frame $(\widetilde{u}_j)_1^\nu$ of $\widetilde{f}$ we have the basis identity

$$\sum_{j=1}^{\nu} |\langle \widetilde{u}_j, x\rangle|^p = \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right) \int |\langle y, x\rangle|^p \, d\sigma(y).$$

By Lemma 4.4.1 we obtain the projective cubature formula of index p with the nodes and the weights

$$x_j = \frac{\widetilde{u}_j}{\|\widetilde{u}_j\|} \in \mathbf{S}(\mathbf{K}^m), \quad \varrho_j = \frac{\|\widetilde{u}_j\|^p}{\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)}, \quad 1 \leq j \leq \nu, \tag{4.310}$$

respectively.

Conversely, let

$$\sum_{j=1}^{\nu} \phi(x_j)\varrho_j = \int \phi \, d\sigma, \quad \phi \in \mathtt{Pol}_{\mathbf{K}}(\mathbf{K}^m; p).$$

In particular, if $\phi(y) = |\langle y, x\rangle|^p$ then we get (4.307) with

$$u_j = \left(\varrho_j \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)\right)^{\frac{1}{p}} x_j, \quad 1 \leq j \leq \nu; \quad u_j = 0, \quad \nu < j \leq n. \tag{4.311}$$

□

Theorem 4.7.1 with some ingredients of its proof allows us systematically apply the theory of projective cubature formulas to the isometric embeddings $\ell_2^m \to \ell_p^n$. In particular, in the process of the proof of Theorem 4.7.1 we also establish a $1-1$ *correspondence between irreducible isometric embeddings $\ell_2^m \to \ell_p^n$ and projective cubature formulas of index p with n nodes on $\mathbf{S}(\mathbf{K}^m)$ up to change of the latter for some projectively equivalent ones.*

PROPOSITION 4.7.2. *Let $(u_1, \ldots, u_r, 0, \ldots, 0)$ be a frame of an isometric embedding $\ell_2^m \to \ell_p^n$. Then for every integer $q \leq p$ the system*

$$u_j^{(q)} = \left(\frac{\Upsilon_{\mathbf{K}}\left(m, \frac{q}{2}\right)}{\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)}\right)^{\frac{1}{q}} \|u_j\|^{\frac{p-q}{q}} u_j, \quad 1 \leq j \leq r; \quad u_j^{(q)} = 0, \quad r < j \leq n, \tag{4.312}$$

is a frame of an isometric embedding $\ell_2^m \to \ell_q^n$.

Proof. Let $(u_j)_1^n$ be a frame of an isometric embedding $\ell_2^m \to \ell_p^n$. For simplicity let us suppose that the embedding is irreducible. Then by (4.310) the nodes and the weights of the corresponding projective cubature formula of index p are

$$x_j = \frac{u_j}{\|u_j\|}, \quad \varrho_j = \frac{\|u_j\|^p}{\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)}, \quad 1 \le j \le r, \tag{4.313}$$

respectively. By Corollary 4.3.6 the same cubature formula is of index q. In this context we obtain (4.312) using (4.311) with $\nu = r$ and with q instead of p.

The result remains in force in the reducible case because of (4.308) and (4.309). □

The next statement follows from (4.310) and the relation

$$\sum_{j=1}^{\nu} \varrho_j = 1,$$

taking the reduction formulas (4.308)and (4.309) into account.

PROPOSITION 4.7.3. *For any isometric embedding $\ell_2^m \to \ell_p^n$ its frame $(u_j)_1^n$ satisfies*

$$\boxed{\sum_{j=1}^{n} \|u_j\|^p = \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)}\ . \tag{4.314}$$

COROLLARY 4.7.4. *If the frame of an isometric embedding $\ell_2^m \to \ell_p^n$ lies on a sphere centered at the origin then this sphere is*

$$\boxed{\|u\| = \left(\frac{\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)}{n}\right)^{\frac{1}{p}}}\ . \tag{4.315}$$

This is just the case the support of the corresponding cubature formula is a projective p-design, cf. (4.315) and (4.313).

Now we substitute $x = u_i$ into (4.307) and get

$$\sum_{j=1}^{n} |\langle u_j, u_i\rangle|^p = \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right) \int |\langle y, u_i\rangle|^p \, d\sigma(y) = \|u_i\|^p, \quad 1 \le i \le n, \tag{4.316}$$

by the Hilbert identity (2.47). By summation over i with taking (4.314) into account, we obtain

PROPOSITION 4.7.5. *The frame $(u_j)_1^n$ of an isometric embedding $\ell_2^m \to \ell_p^n$ satisfies*

$$\sum_{i,j=1}^{n} |\langle u_j, u_i\rangle|^p = \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right). \tag{4.317}$$

The identities (4.314) and (4.317) are extremal cases for the following

THEOREM 4.7.6. *The frame* $(u_j)_1^n$ *of an isometric embedding* $\ell_2^m \to \ell_p^n$ *satisfies*

$$\boxed{\sum_{i,j=1}^{n} |\langle u_j, u_i\rangle|^q \cdot \|u_j\|^{p-q} \cdot \|u_i\|^{p-q} = \frac{\left(\Upsilon_{\mathbf{K}}(m, \frac{p}{2})\right)^2}{\Upsilon_{\mathbf{K}}\left(m, \frac{q}{2}\right)}, \quad 0 \le q \le p, \quad q \equiv 0 (\mathrm{mod}\ 2)} \ . \tag{4.318}$$

Conversely, if a system $(u_j)_1^n$ *satisfies (4.318) then this is a frame of an isometric embedding* $\ell_2^m \to \ell_p^n$.

Proof. For simplicity we only consider the case of irreducible system $(u_j)_1^n$. Then (4.313) defines a measure ϱ on the unit sphere. The relations (4.318) are nothing else than (4.92) with $t = \frac{p}{2}$, $k = \frac{q}{2}$ applying to the measure ϱ. For this reason Theorem 4.7.6 follows from Corollary 4.3.10 in both directions.. □

Theorem 4.7.6 characterizes the frame of an isometric embedding $f : \ell_2^m \to \ell_p^n$ in terms of its Gram matrix, which is a complete unitary invariant in view of Witt's Theorem. In this context let us note that, obviously, if $(u_k)_1^n$ is a frame of f then for any unitary operator $V : \ell_2^m \to \ell_2^m$ the system $(Vu_k)_1^n$ is also a frame of f.

The following result is a direct consequence of Theorem 4.7.1 per se.

THEOREM 4.7.7. *Given* m *and* p*, an isometric embedding* $\ell_2^m \to \ell_p^n$ *does exist if and only if*

$$n \ge N_{\mathbf{K}}(m, p). \tag{4.319}$$

Proof. By Theorem 4.7.1 a projective cubature formula of index p with $n = N_{\mathbf{K}}(m,p)$ nodes on $\mathbf{S}(\mathbf{K}^m)$ provides an isometric embedding $\ell_2^m \to \ell_p^n$ with any $n \ge N_{\mathbf{K}}(m,p)$. Conversely, if there exists an isometric embedding $\ell_2^m \to \ell_p^n$ then (4.319) immediately follows from Theorem 4.7.1. □

COROLLARY 4.7.8. *Given* m *and* p*, the minimal number* n *such that there exists an isometric embedding* $\ell_2^m \to \ell_p^n$ *is* $N_{\mathbf{K}}(m,p)$*, the minimal number of nodes of a projective cubature formula of index* p *on* $\mathbf{S}(\mathbf{K}^m)$.

DEFINITION 4.7.9. (cf. Definition 4.4.3) *Given* m *and* p*, an isometric embedding* $\ell_2^m \to \ell_p^n$ *is called* **minimal** *if* $n = N_{\mathbf{K}}(m,p)$*, i.e. there is no any isometric embedding* $\ell_2^m \to \ell_p^\nu$ *with* $\nu < n$.

REMARK 4.7.10. *A minimal isometric embedding is irreducible. An irreducible isometric embedding is minimal if and only if so is the corresponding cubature formula.*

Thus, all bounds for the number of nodes of projective cubature formulas are automatically true for the corresponding isometric embeddings. In particular, lower bound (4.96) provides the same lower bound for the isometric embeddings.

THEOREM 4.7.11. *If an isometric embedding* $\ell_2^m \to \ell_p^n$ *exists then*

$$n \ge \Lambda_{\mathbf{K}}(m, \frac{p}{2}). \tag{4.320}$$

In a more detail

$$n \geq \begin{cases} \binom{m+\frac{p}{2}-1}{m-1} & (\mathbf{K}=\mathbf{R}) \\ \binom{m+[\frac{p}{4}]-1}{m-1} \cdot \binom{m+[\frac{p+2}{4}]-1}{m-1} & (\mathbf{K}=\mathbf{C}) \\ \frac{1}{2m-1}\binom{2m+[\frac{p}{4}]-2}{2m-2} \cdot \binom{2m+[\frac{p+2}{4}]-1}{2m-2} & (\mathbf{K}=\mathbf{H}), \end{cases} \tag{4.321}$$

see (4.97).

DEFINITION 4.7.12. (cf. Definition 4.3.17) *An isometric embedding $\ell_2^m \to \ell_p^n$ is called* **tight** *if the equality is obtained in (4.321).*

REMARK 4.7.13. *An irreducible isometric embedding is tight if and only if so is the corresponding cubature formula. Any tight isometric embedding is minimal, a fortiori, it is irreducible.* The converse is not true. For example, combining Corollary 4.7.8 and Theorem 4.6.25 we see that the *minimal isometric embedding $\ell_2^2 \to \ell_8^{10}$ over $\mathbf{C}$ is not tight.* $\square$

THEOREM 4.7.14. *Let*

$$\boxed{\mathcal{R}_{\mathbf{K}}(m,p) = \left(\frac{\Upsilon_{\mathbf{K}}\left(m,\frac{p}{2}\right)}{\Lambda_{\mathbf{K}}\left(m,\frac{p}{2}\right)}\right)^{\frac{1}{p}}} \;. \tag{4.322}$$

If an isometric embedding $f: \ell_2^m \to \ell_p^n$ is tight then

(i) *its frame $(u_j)_1^n$ lies on a sphere $\mathbf{S}_{\mathbf{K}}(m,p)$ of radius $\mathcal{R}_{\mathbf{K}}(m,p)$ centered at origin;*

(ii) *with $\varepsilon = \varepsilon_{\frac{p}{2}}$ the polynomial*

$$(1+u)^{\varepsilon} P_{[\frac{p}{4}]}^{(\alpha+1,\beta+\varepsilon)}(u) \tag{4.323}$$

annihilates the angle set of the normalized frame

$$\widehat{u}_j = (\mathcal{R}_{\mathbf{K}}(m,p))^{-1} u_j, \quad 1 \leq j \leq n.$$

Conversely, with

$$n = \Lambda_{\mathbf{K}}\left(m, \frac{p}{2}\right), \tag{4.324}$$

let a system $(u_j)_1^n \subset \mathbf{K}^m$ lie on the sphere $\mathbf{S}_{\mathbf{K}}(m,p)$ and (**ii**) *holds. Then $(u_j)_1^n$ is the frame of a tight isometric embedding.*

Proof. Let f be tight. Then it is irreducible and the corresponding projective cubature formula is defined by (4.310) with $\nu = n$ and $\widetilde{u}_j = u_j,\ 1 \leq j \leq n$. Since this cubature formula is tight as well, the weights ϱ_j must be equal $\frac{1}{n}$ according to Theorem 4.3.18. Then (4.310) implies $\|u_j\| = \mathcal{R}_{\mathbf{K}}(m,p),\ 1 \le j \le n$, cf. (4.315). After that (**ii**) immediately follows from Theorem 4.3.18(**ii**).

In the converse direction Theorem 4.3.18 implies that $(\widehat{u}_j)_1^n$ is a tight projective p-design on $\mathbf{S}(\mathbf{K}^m)$. Then by (4.311) and (4.324) we obtain an isometric embedding with the frame $\mathcal{R}_{\mathbf{K}}(m,p)(\widehat{u}_j)_1^n = (u_j)_1^n$. This isometric embedding is tight because of (4.324). □

In conclusion we give a list of isometric embeddings $\ell_2^m \to \ell_p^n$ which follows from the results concerning cubature formulas.

THEOREM 4.7.15. *There exist some isometric embeddings $\ell_{2;\mathbf{K}}^m \to \ell_{p;\mathbf{K}}^n$ corresponding to the triples (m,p,n) from the following tables*

$\mathbf{K} = \mathbf{R}:$

m	2	3	7	8	23	23	24
p	p	4	4	6	4	6	10
n	$\frac{p}{2}+1$	6	28	120	276	2300	98280

, (4.325)

$\mathbf{K} = \mathbf{C}:$

m	2	2	2	3	4	6	8
p	4	6	10	4	6	6	4
n	4	6	12	9	40	126	64

, $\mathbf{K} = \mathbf{H}:$

m	2	5
p	6	6
n	10	165

(4.326)

All these embeddings are tight.

Proof follows from Theorem 4.7.1 and the relevant statements from Corollary 4.4.11, the tables (4.170), (4.182), (4.183), Corollary 4.6.16 and, finally, the table (4.188).□

Now we combine Theorems 4.7.1 and 4.4.7. This yields

THEOREM 4.7.16. *Each isometric embedding $\ell_{2;\mathbf{R}}^{\delta(m-1)} \to \ell_{p;\mathbf{R}}^n$ generates an isometric embedding $\ell_{2;\mathbf{K}}^m \to \ell_{p;\mathbf{K}}^N$ where*

$$N = \left(\frac{p}{2}+1\right) N_{\mathbf{R}}\left(\delta, 2\left[\frac{p}{4}\right]\right)\nu, \quad \nu \le n. \tag{4.327}$$

Note that $\nu = n$ as soon as the initial isometric embedding is irreducible. Moreover, in this case the resulting isometric embedding is irreducible as well.

On the base of Theorem 4.7.16 we obtain the following chain of results.

THEOREM 4.7.17. (cf. Theorem 4.4.9) *Assume that for given m,p there exists an isometric embedding $\ell_{2;\mathbf{R}}^m \to \ell_{p;\mathbf{R}}^n$. Then there exists an isometric embedding $\ell_{2;\mathbf{R}}^m \to \ell_{p;\mathbf{R}}^N$ with*

$$N = \left(\frac{p}{2}+1\right)\nu, \quad \nu \le n. \tag{4.328}$$

THEOREM 4.7.18.(cf. Corollary 4.4.12) *Assume that for given m,p there exists an isometric embedding $\ell_{2;\mathbf{R}}^m \to \ell_{p;\mathbf{R}}^n$. Then there exists an isometric embedding $\ell_{2;\mathbf{R}}^M \to \ell_{p;\mathbf{R}}^N$ with*

$$N = \left(\frac{p}{2}+1\right)^{M-m}\nu \tag{4.329}$$

where $\nu \le n$, $M \ge m$.

THEOREM 4.7.19.(cf. Theorem 4.4.16) *Assume that for given m, p there exists an isometric embedding $\ell_{2;\mathbf{R}}^{2m} \to \ell_{p;\mathbf{R}}^{n}$. Then there exists an isometric embedding $\ell_{2;\mathbf{C}}^{m+1} \to \ell_{p;\mathbf{C}}^{N}$ with*

$$N = \left(\frac{p}{2}+1\right)\left(\left[\frac{p}{4}\right]+1\right)\nu, \quad \nu \le n. \tag{4.330}$$

THEOREM 4.7.20.(cf. Theorem 4.4.20) *Assume that for given m, p there exists an isometric embedding $\ell_{2;\mathbf{R}}^{4m} \to \ell_{p;\mathbf{R}}^{n}$. Then there exists an isometric embedding $\ell_{2;\mathbf{H}}^{m+1} \to \ell_{p;\mathbf{H}}^{N}$ with*

$$N = \begin{cases} L(\frac{p}{2})\nu, & 4 \le \frac{p}{2} < 40, \quad p \ne 24, 26 \\ (\frac{p}{2}+1)\left(\left[\frac{p}{4}\right]+1\right)^3 \nu, & p \ge 40 \text{ or } p = 24, 26 \end{cases} \tag{4.331}$$

and $\nu \le n$.

Finally, we collect the non-tight isometric embeddings $\ell_2^m \to \ell_p^n$ following from the projective cubature formulas obtained or quoted in Sections 4.4 and 4.6.

K = R

m	3	3	4	4	4	4	4	4	4	5	5	5	5	5
p	6	8	4	6	8	10	14	16	18	4	6	8	10	14
n	11	16	11	24	60	60	360	360	360	16	96	300	360	2880

, (4.332)

m	5	5	6	6	6	7	7	8	9	10	11	16
p	16	18	4	6	10	4	6	6	6	6	6	6
n	3420	3600	22	63	2160	28	113	120	480	1920	7680	2160

, (4.333)

m	17	18	23	24	24	25	25	25	26
p	6	6	4	6	8	6	8	10	4
n	8640	34560	276	9200	98280	36800	491400	589680	147200

, (4.334)

m	26	26	27	27	28
p	8	10	6	10	10
n	2457000	3538080	588800	21228480	127370880

. (4.335)

K = C

m	2	2	2	2	2	3	4	5	5	9	9	10
p	8	12	14	16	18	6	4	4	6	4	6	14
n	10	22	60	60	60	21	20	45	960	90	17280	63685440

, (4.336)

m	12	12	12	13	13	13
p	6	8	10	6	8	10
n	32760	32760	32760	73600	1474200	1769040

. (4.337)

K = H

m	2	3	3	3	3	4	4	5	7
p	4	4	6	8	10	4	6	4	10
n	10	63	63	315	315	36	180	165	6486480

. (4.338)

An additional information related to $p = 4$ follows from Theorems 4.4.23, 4.4.27 and 4.4.31.
The concrete results contained in the tables (4.332) − (4.338) are the best known for us at present.

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