Algebra Qualifying Examination
August 6, 2019

Instructions:

- Read all problems first; make sure that you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit awarded will be based on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. Do scratch work on a separate page.
- Start each problem on a new page, clearly marking the problem number on that page.
- Rings always have an identity and all modules are left modules.
- Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.

1. Let G be a group of order 91. Prove that G is abelian. (Note that $91 = 7 \cdot 13$.)

2. Let G be a group and let $Z(G)$ be the center of G. Let $n = [G : Z(G)]$.
   (a) Prove that every conjugacy class of G has at most n elements.
   (b) Suppose $n > 1$. Is there an example of a group G with $[G : Z(G)] = n$ and an element $g \in G$ such that the conjugacy class of g has *exactly* n elements? Justify your answer.

3. Let R be a ring. Let N be the subset of R consisting of all nilpotent elements. (An element $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer n.)
   (a) Prove that if R is commutative, then N is an ideal.
   (b) If R is not commutative, must N be an ideal? Prove or give a counterexample.

4. Let R be a finite ring. Prove that if R has no zero divisors, then R is a division ring (that is, each nonzero element of R is invertible).

5. For the following questions, A is a 3×3 matrix with entries in $\mathbb{C}$ and I is the 3×3 identity matrix.
   (a) List all possible 3×3 matrices A in Jordan canonical form having 5 as the only eigenvalue.
   (b) Which of the matrices A from part (a) satisfy $\dim(\ker(A - 5I)) = 2$?
   (c) Let $V = \mathbb{C}^3$ and let A be any of the matrices from part (a). Consider V to be a $\mathbb{C}[x]$-module via $p(x) \cdot v = p(A)v$ for all $v \in V$, $p(x) \in \mathbb{C}[x]$. For which of the matrices A from part (a) is V a cyclic $\mathbb{C}[x]$-module?

6. Let R be a ring, and let M be an R-module. Prove that the following conditions are equivalent:
   (i) Every R-submodule N of M is finitely generated.
   (ii) M satisfies the ascending chain condition, that is for every sequence of R-submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

   of M, there is a positive integer t such that $M_s = M_t$ for all $s \geq t$.

7. (a) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} G = 0$ for all finite abelian groups G.
   (b) Find $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$. Justify your answer.

8. Let $f(x) = x^4 - 4$ in $\mathbb{Q}[x]$.
   (a) Find the splitting field K of f over $\mathbb{Q}$.
   (b) Find the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

9. Let K be a field extension of F such that $K = F(\alpha, \beta)$ for elements α, β of K. Suppose $[F(\alpha) : F] = m$ and $[F(\beta) : F] = n$ for some positive integers m, n.
   (a) Prove that if m, n are relatively prime, then $[K : F] = mn$.
   (b) Does the conclusion of (a) necessarily hold in the absence of the relatively prime hypothesis? Prove or give a counterexample.