Algebra Qualifying Examination
August 6, 2020

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed next to each problem number.
- Read all problems first. Make sure that you understand them, and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit awarded for your answers will be based on the correctness of your answers, as well as the clarity and main steps of your reasoning. Answers must be legible and written in a structured and understandable manner. Do scratch work on a separate page.
- Start each problem on a new page, clearly marking the problem number and your name on that page.
- You may use a calculator to check your computations (but it may not be used as a step in your reasoning).

Notation: Throughout, let $\mathbb{Z}$ denote the ring of integers; let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively. For a set S, we let $|S|$ denote its cardinality. All non-zero rings are assumed to have a multiplicative identity $1 \neq 0$. For a Galois extension of fields L/K, let $\mathrm{Gal}(L/K)$ denote its Galois group.

1. [12 points] Let p be a prime number, and let G be a finite group of order p^n for some $n \geq 1$. Suppose G acts on a finite set X, and let

$$Y = \{x \in X : \forall\, \sigma \in G,\ \sigma \cdot x = x\}.$$

   You may recall a lemma that states that $|X| \equiv |Y| \pmod{p}$. Prove that this is true.

2. [12 points] Let G be a group of order $520 = 2^3 \cdot 5 \cdot 13$.
   (a) For $p = 2,\ 5,\ 13$, let N_p denote the number of Sylow p-subgroups of G. According to the Sylow Theorems, what are the possibilities for N_2, N_5, and N_{13}?
   (b) If $N_5 \neq 1$, how many elements are there of order 5 in G? If $N_{13} \neq 1$, how many elements are there of order 13 in G?
   (c) Prove that G is not simple.

3. [12 points; (a) 8 pts., (b) 4 pts.] Let R be a commutative ring, and let $M \subseteq R$ be an ideal of R. We let M^2 be the ideal of R generated by the set $\{ab : a,\, b \in M\}$.
   (a) Prove the following statement. If M is both maximal and principal, then for any ideal $I \subseteq R$ with $M^2 \subseteq I \subseteq M$, it must be the case that $I = M$ or $I = M^2$. (Hint: define and make use of an R-module homomorphism $\phi : R/M \to M/M^2$.)
   (b) Give examples that show that neither of the conditions on M in part (a) can be removed.

4. [12 points] Let R be a unique factorization domain. Prove the following.

   (a) Every non-zero element of R is contained in only finitely many principal ideals of R.

   (b) For any infinite chain of principal ideals in R,
   $$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots \subseteq (a_n) \subseteq \cdots \subseteq R,$$
   there exists a positive integer s so that for all $t \geq s$, we have $(a_t) = (a_s)$.

5. [12 points] Let K be a field, let V be a finite dimensional K-vector space, and let $V^* = \mathrm{Hom}_K(V, K)$ be the dual K-vector space of V. For a K-subspace $W \subseteq V$ define the annihilator
   $$\mathrm{Ann}(W) := \{\lambda \in V^* : \forall\, w \in W \text{ we have } \lambda(w) = 0\}.$$
   Prove that the following statement holds: If U, W are K-subspaces of V with $U \subseteq W$, then $\mathrm{Ann}(U)/\mathrm{Ann}(W)$ and $(W/U)^*$ are isomorphic as K-vector spaces. Furthermore, provide an explicit isomorphism.

6. [12 points; (a) 4 pts., (b) 8 pts.] Let R be a commutative ring, and let P and M be R-modules.

   (a) Define
   $$h : \mathrm{Hom}_R(P, R) \times M \to \mathrm{Hom}_R(P, M)$$
   by $h(f, m)(p) = f(p)m$ for $f \in \mathrm{Hom}_R(P, R)$, $m \in M$, and $p \in P$. Show that h is R-bilinear.

   (b) Let
   $$\phi : \mathrm{Hom}_R(P, R) \otimes_R M \to \mathrm{Hom}_R(P, M)$$
   be the function induced by the R-bilinear map h in part (a). Prove that if P is finitely generated and projective as an R-module, then ϕ is an R-module isomorphism. (Hint: first do the case when P is free.)

7. [12 points] Let $f(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + 1 \in \mathbb{Q}[x]$ be irreducible, and let E be the splitting field of f contained in $\mathbb{C}$. The reciprocal polynomial of f is the polynomial $g(x) := x^6 f(1/x)$. Now we know that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup of S_6. If $f = g$, then prove that $\mathrm{Gal}(E/\mathbb{Q})$ is **not** isomorphic to all of S_6.

8. [16 points] Consider the real number $\alpha = \sqrt{2 + \sqrt{2}}$.

   (a) What is the irreducible polynomial f of α over $\mathbb{Q}$? (Be sure to prove that the polynomial you find is irreducible.)

   (b) Determine the splitting field $E \subseteq \mathbb{C}$ of f over $\mathbb{Q}$. What is $[E : \mathbb{Q}]$?

   (c) Determine the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ and determine how each automorphism in G acts on the roots of f.

   (d) For each subgroup $H \subseteq G$, what is the fixed field E^H?