Algebra Qualifying Examination
12 January 2010

Instructions:

- There are eight problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- All rings have an identity and all modules are unital.

Notation: Throughout $\mathbb{Q}$ denotes the rationals and $Aut(K/\mathbb{Q})$ denotes the group of field automorphisms of K fixing $\mathbb{Q}$.

1. (21 points) The *exponent* $exp(G)$ of a group G is the smallest $k \in \{1, 2, \ldots\} \cup \{\infty\}$ such that $g^k = e$ for all $g \in G$.
   (a) Show that a finitely generated abelian group A with $exp(A) < \infty$ is finite.
   (b) Give an example of an infinite group of finite exponent.
   (c) Give an example of a group G in which every element has finite order but $exp(G) = \infty$.
2. (18 points) Let $T \in \mathrm{End}(V)$ be a linear operator on a vector space V with $\dim(V) = n$ such that $min_T(x) = char_T(x)$, i.e. the minimal and characteristic polynomials of T coincide.
   (a) Show that there exists an $\alpha \in V$ such that $\{\alpha, T(\alpha), \ldots, T^{n-1}(\alpha)\}$ is a basis for V.
   (b) Show that if $U \in \mathrm{End}(V)$ satisfies $UT = TU$ then U is a polynomial in T.
3. (8 points) Let R be a principal ideal domain. Show that if $P \neq (0)$ is a prime ideal then P is maximal.
4. (10 points) An R-module M is *indecomposable* if there are no R-submodules $A \neq 0$ and $B \neq 0$ of M such that $M = A \oplus B$. Show that if R is a principal ideal domain then if M is indecomposable and finitely generated then either $M \cong R$ or $M \cong R/(p^n)$ for some prime element p of R.
5. (7 points) Show that a group of order 80 cannot be simple.
6. (21 points) Find the degree of the splitting field (over $\mathbb{Q}$) of the following polynomials:
   (a) $x^3 - x - 2$
   (b) $(x^2 - 2)(x^2 - 5)$
   (c) $(x^2 - 2)(x^2 - 5)(x^2 - 10)$

7. (7 points) Let G be a finite group and H a Sylow p-subgroup of G. Show that $N_G(N_G(H)) = N_G(H)$ where

$$N_G(K) := \{g \in G : gKg^{-1} = K\}.$$

(You may use the Sylow theorems, but you may **not** state a theorem from a text that is identical to the content of the problem).

8. (8 points) Determine the Galois group of $x^5 - 6x + 3$ over $\mathbb{Q}$ (justify your answer).