Applied/Numerical Analysis Qualifying Exam

August 13, 2014

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 13, 2014

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let f be a 2π-periodic function.

(a) Sketch a proof of the following: *If f is a piecewise $C^{(1)}$ (i.e., can have jumps), and if $S_N = \sum_{n=-N}^{N} c_n e^{inx}$ is the N^{th} partial sum of the Fourier series for f, then, for every $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} S_N(x) = \frac{f(x^+) + f(x^-)}{2}.$$

(b) Show that if f is $C^{(1)}$, then the convergence is uniform.

Problem 2. Consider the boundary value problem

$$u'' = f,\ u(0) - u'(0) = 0,\ u(1) + u'(1) = 0. \tag{2.1}$$

(a) Find the Green's function, $G(x, y)$, for (2.1).

(b) Show that $Gf(x) = \int_0^1 G(x,y)f(y)dy$ is compact and self adjoint on $L^2[0, 1]$.

(c) State the spectral theorem for compact, self-adjoint operators. Use it to show that the (normalized) eigenfunctions of the eigenvalue problem $u'' + \lambda u = 0$, $u(0) - u'(0) = 0$, $u(1) + u'(1) = 0$ form a complete orthonormal set in $L^2[0, 1]$. (Hint: How are the eigenfunctions of G related to those of $u'' + \lambda u = 0$, $u(0) - u'(0) = 0$, $u(1) + u'(1) = 0$?)

Problem 3. Let $k(x,y) = x^2y^3$, $Ku(x) = \int_0^1 k(x,y)u(y)dy$, and $Lu = u - \lambda Ku$.

(a) Show that L has closed range.

(b) Determine the values of λ for which $Lu = f$ has a solution for all f. Solve $Lu = f$ for these values of λ.

(c) For the remaining values of λ, find a condition on f that guarantees a solution to $Lu = f$ exists. When f satisfies this condition, solve $Lu = f$.

Problem 4. Let $p \in C^{(2)}[0,1]$, and $q, w \in C[0,1]$, with $p, q, w > 0$. Consider the Sturm-Liouville (SL) eigenvalue problem, $(p\phi')' - q\phi + \lambda w\phi = 0$, subject to $\phi(0) = 0$ and either (A) $\phi(1) = 0$ or (B) $\phi'(1) + \phi(1) = 0$. In addition, for $\phi \in C^{(1)}[0,1]$, let $D[\phi] := \int_0^1 (p\phi'^2 + q\phi^2)dx$ and $H[\phi] := \int_0^1 w\phi^2 dx$.

(a) Show that minimizing the functional $D[\phi]$, subject to the constraint $H[\phi] = 1$ and boundary conditions $\phi(0) = \phi(1) = 0$, yields the SL problem (A).

(b) State the variational problem that will yield the SL problem (B). Verify that your answer is correct by calculating the variational (Fréchet) derivative and setting it equal to 0.

(c) State the Courant MINIMAX Principle. (Eigenvalues increase: $\lambda_1 < \lambda_2 < \lambda_3 \cdots$.) Use it to show that the n^{th} eigenvalue of the SL problem (A) is larger than or equal to the n^{th} eigenvalue of the SL problem (B).

APPLIED MATH QUALIFIER: NUMERICAL ANALYSIS PART

August 13, 2014

Problem 1. Let $K = [0,1]^2$ be the unit square and denote by q_i, $i = 1, ..., 4$, its vertices and by a_i, $i = 1, .., 4$, the midpoints of its sides. Set $P = \mathbb{Q}^1 := \{p(x,y) = (ax+b)(cy+d) \ : \ a,b,c,d \in \mathbb{R}\}$ be the space of polynomial of degree at most 1 in each direction.

(1) For $\Sigma := \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where $\sigma_i(p) = p(q_i)$, $i = 1, ..., 4$, show that the finite element triplet (K, P, Σ) is unisolvent.
(2) For $\widetilde{\Sigma} := \{\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3, \widetilde{\sigma}_4\}$, where $\widetilde{\sigma}_i(p) = p(a_i)$, $i = 1, ..., 4$, show that the finite element triplet $(K, P, \widetilde{\Sigma})$ is **not** unisolvent.

Problem 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex polygonal domain. Let $\mathbb{V} := H_0^1(\Omega)$ with inner product and corresponding norm

$$(u,v)_1 := D(u,v) + (u,v) \qquad \text{and} \qquad \|u\|_1 := (u,u)_1^{1/2},$$

respectively, where

$$(u,v) := \int_\Omega u\, v\, dx \qquad \text{and} \qquad D(u,v) := \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx.$$

For any positive constant k, define on $\mathbb{V} \times \mathbb{V}$ the form

$$a_k(u,v) := D(u,v) - k(u,v).$$

(1) Show that there exists a $k_0 > 0$ such that $a_k(.,.)$ is continuous and coercive on $\mathbb{V}$ for $k \le k_0$.
(2) Let $f \in L^2(\Omega)$. Show that for $k \le k_0$ there exists a unique function $u \in \mathbb{V}$ such that

$$a_k(u,v) = (f,v), \qquad \forall v \in \mathbb{V}.$$

(3) Let $\mathbb{V}_h$ be a subspace of $\mathbb{V}$ and h be a mesh parameter. The Galerkin approximation $u_h \in \mathbb{V}_h$ satisfies

$$a_k(u_h, v_h) = (f, v_h), \qquad \forall v_h \in \mathbb{V}_h.$$

Assume that $\mathbb{V}_h$ has the following approximation property: There exists a constant C independent of h such that for all $v \in H^2(\Omega)$ there holds

$$\inf_{v_h \in \mathbb{V}_h} \|v - v_h\|_1 \le Ch\|v\|_2,$$

where $\|.\|_2$ is the natural norm on $H^2(\Omega)$. Prove Cea's lemma in this context and deduce the existence of a constant independent of h and u such that

$$\|u - u_h\|_1 \le Ch\|u\|_2,$$

provided that $u \in H^2(\Omega)$.
(4) Use a duality argument to derive an optimal L^2-norm estimate for the error using the previous result. You can use without proof that there exists a constant C such that for any $g \in L^2(\Omega)$, the unique solution $w \in \mathbb{V}$ of

$$a_k(w,v) = (g,v) \qquad \forall v \in \mathbb{V}$$

belongs to $H^2(\Omega)$ and

$$\|w\|_2 \le C\|g\|_0.$$

Problem 3. Let Ω be a bounded polygonal domain. Let $T > 0$ be a given final time, f be a given real valued function in $C^0(\overline{\Omega} \times [0,T])$, and let u_0 be a given real valued function in $H^1(\Omega)$. Consider the parabolic PDE

$$\begin{aligned} \frac{\partial u}{\partial t}(\mathbf{x},t) - \Delta u(\mathbf{x},t) &= f(\mathbf{x},t) && \text{in} \quad \Omega \times (0,T), \\ u(\mathbf{x},t) &= 0 && \text{on} \quad \partial\Omega \times (0,T), \\ u(\mathbf{x},0) &= u_0(\mathbf{x}) && \text{in} \quad \Omega. \end{aligned}$$

We focus on a *second order* semi-discretization in time. Accept as a fact that the above parabolic problem has one and only one solution that is sufficiently smooth and satisfies for all $v \in H_0^1(\Omega)$

$$\int_\Omega \frac{\partial u}{\partial t}(\mathbf{x},t)v(\mathbf{x})\, d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x},t)\cdot\nabla v(\mathbf{x})\, d\mathbf{x} = \int_\Omega f(\mathbf{x},t)v(\mathbf{x})\, d\mathbf{x}$$

and $u(0,\mathbf{x}) = u_0(\mathbf{x})$ a.e. in Ω.

(1) Let $N \geq 2$ be an integer, set $\tau := T/N$, $t_n := n\tau$ for $0 \leq n \leq N$, and

$$f^{n-1/2}(\mathbf{x}) := \frac{1}{2}\left(f(\mathbf{x},t_{n-1}) + f(\mathbf{x},t_n)\right).$$

Then, starting from $u^0 = u_0$, consider the following problem: For each $1 \leq n \leq N$, given $u^{n-1} \in H_0^1(\Omega)$ find $u^n \in H_0^1(\Omega)$ satisfying for any $v \in H_0^1(\Omega)$

$$\frac{1}{\tau}\int_\Omega (u^n(\mathbf{x}) - u^{n-1}(\mathbf{x}))v(\mathbf{x})d\mathbf{x} + \int_\Omega \nabla(\frac{u^n(\mathbf{x}) + u^{n-1}(\mathbf{x})}{2})\cdot\nabla v(\mathbf{x})\, d\mathbf{x}$$
$$= \int_\Omega f^{n-1/2}(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}.$$

Prove that the above problem has one and only one solution $u^n \in H_0^1(\Omega)$.

(2) Show that for any $n = 1,\dots,N$ there holds

$$\|u^n\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{i=1}^n \tau\|\nabla(\frac{u^i + u^{i-1}}{2})\|^2_{L^2(\Omega)} \leq \|u^0\|^2_{L^2(\Omega)} + \frac{C_\Omega^2}{2}\sum_{i=1}^n \tau\|f^{i-1/2}\|^2_{L^2(\Omega)},$$

where C_Ω is the Poincaré constant.

(3) Show that for all $v \in H_0^1(\Omega)$

$$\frac{1}{\tau}\int_\Omega (u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1}))v(\mathbf{x})\, d\mathbf{x} + \int_\Omega \nabla\left(\frac{u(\mathbf{x},t_n) + u(\mathbf{x},t_{n-1})}{2}\right)\cdot\nabla v(\mathbf{x})\, d\mathbf{x}$$
$$= \int_\Omega f^{n-1/2}(\mathbf{x})v(\mathbf{x})\, d\mathbf{x} + \int_\Omega E^{n-1/2}(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x},$$

where

$$E^{n-1/2}(\mathbf{x}) := \frac{1}{\tau}(u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1})) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(\mathbf{x},t_n)) + \frac{\partial u}{\partial t}(\mathbf{x},t_{n-1}))\right).$$

(4) Use the Taylor expansion formula

$$f(s) = f(a) + f'(a)(s-a) + \frac{1}{2}f''(a)(s-a)^2 + \frac{1}{2}\int_a^s (s-t)^2 f'''(t)dt$$

and similar formula for the derivative to deduce the following bound for $E^{n-1/2}$

$$\|E^{n-1/2}\|^2_{L^2(\Omega)} \leq C\tau^3\int_{t_{n-1}}^{t^n}\int_\Omega \left|\frac{\partial^3}{\partial t^3}u\right|^2 dx dt,$$

where C is a constant independent of N and u.

(5) Denote the errors by $e^n(\mathbf{x}) := u(.,t_n) - u^n(.)$, $n = 1,\dots,N$, and prove *using the results obtained in the previous steps* that there exists a constant C independent of N and u such that

$$\left(\sup_{1\leq n\leq N}\|e^n\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{n=1}^N \tau\|\nabla(\frac{e^n + e^{n-1}}{2})\|^2_{L^2(\Omega)}\right)^{1/2} \leq C\tau^2\left(\int_0^T\int_\Omega \left|\frac{\partial^3}{\partial t^3}u\right|^2 dx dt\right)^{1/2}.$$