APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER
Applied Analysis Part, 2 hours
August 8, 2018

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let H be the Sobolev space $H = H_0^1[0,1] := \{f \in L^2[0,1] : f' \in L^2[0,1] \text{ and } f(0) = f(1) = 0\}$, where $\langle f, g\rangle_H := \int_0^1 f'(x)g'(x)dx$. You may assume that H is a Hilbert space and that if $f \in H$, then $f \in C[0,1]$.

(a) State and prove the Riesz Representation Theorem.
(b) Show that if $f \in H$, then $\|f\|_{C[0,1]} \le \|f\|_H$.
(c) Let $f \in C[0,1]$ and let the "delta" functional $\delta_x(f) = f(x)$. Use parts (a) and (b) to show that there exists $G(x,\cdot) \in H$ such that $f(x) = \langle f, G(x,\cdot)\rangle_H$ for all $f \in H$. Verify that $G(x,y) = G(y,x)$. (Hint: Let $f(\cdot) = G(y,\cdot)$.)
(d) Let $X := \{x_1 < x_2 < \cdots < x_n\} \subset \mathbb{R}$. Define the $n \times n$ matrix A by $A_{j,k} := G(x_j, x_k)$. If A is invertible, show that there exists a unique $s(x) = \sum_{k=1}^n c_k G(x, x_k) \in H$ such that s interpolates f on X – i.e., $s(x_j) = f(x_j)$, $j = 1, \ldots, n$.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Let $\phi \in \mathcal{D}$ and define $\phi_h(x) := (\phi(x) - \phi(x-h))/h$. Show that, in the sense of $\mathcal{D}$, $\lim_{h\to 0}\phi_h = \phi'$. (Hint: apply Taylor's formula to $\phi_h^{(n)} - \phi^{(n+1)}$.)
(c) Let $T \in \mathcal{D}'$ and define $T_h = (T(x+h) - T(x))/h$. Show that, in the sense of distributions, $\lim_{h\to 0} T_h = T'$.
(d) Use (c) to show that if $T(x) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0, \end{cases}$ then $T'(x) = -\delta(x)$.

Problem 3. Let $k(x,y) = x^4 y^{12}$ and consider the operator $Ku(x) = \int_0^1 k(x,y)u(y)dy$.

(a) Show that K is a Hilbert-Schmidt operator and that $\|K\|_{\text{op}} \le 1/15$.
(b) State the Fredholm Alternative for the operator $L = I - \lambda K$. Explain why it applies in this case. Find all values of λ such that $Lu = f$ has a unique solution for all $f \in L^2[0,1]$.
(c) Use a Neumann series to find the resolvent $(I - \lambda K)^{-1}$ for λ small. Sum the series to find the resolvent.

Problem 4. Consider the one dimensional heat equation, $u_t = u_{xx}$, with $u(x,0) = f(x)$, where $-\infty < x < \infty$ and $0 < t < \infty$. Use Fourier transforms to show that the solution $u(x,t)$ is given by

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t)f(y)dy, \quad K(x-y,t) = (4\pi t)^{-1/2}e^{-(x-y)^2/4t}.$$

You may use any consistent Fourier transform convention[1] and any of the standard Fourier transform properties. You are also given that $\int_{-\infty}^{\infty} e^{-u^2 \pm ivu}du = \sqrt{\pi}e^{-v^2/4}$.

[1]For example, $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$ & $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{ix\xi}dx$, or $\widehat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\xi}dx$ & $f(x) = \int_{-\infty}^{\infty}\widehat{f}(\xi)e^{-ix\xi}dx$.

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Numerical Analysis Part, 2 hours
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Problem 1. Let $\Omega := (0,1)^2$ and $u \in H^1_{\#}(\Omega) := \{u \in H^1(\Omega) \; : \; \int_\Omega u = 0\}$ be such that

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv =: L(v), \qquad \forall v \in H^1_{\#}(\Omega),$$

where $f \in L^2(\Omega)$ is a given function satisfying $\int_\Omega f = 0$. Accept as a fact that there exists a unique weak solution $u \in H^1_{\#}(\Omega)$.

The goal of this exercise is to analyze a non-conforming finite element method relaxing the vanishing mean value condition.

(1) Consider the approximate problem: Given $0 < \epsilon \leq 1$, seek $u^\epsilon \in H^1(\Omega)$ such that

$$a_\epsilon(u^\epsilon, v) := \int_\Omega \nabla u^\epsilon \cdot \nabla v + \epsilon \int_\Omega u^\epsilon v = L(v), \qquad \forall v \in H^1(\Omega).$$

(i) Show that the above problem has a unique solution; (ii) show that $\int_\Omega u^\epsilon = 0$, i.e. $u^\epsilon \in H^1_{\#}(\Omega)$; (iii) Show that there exists a constant C independent of ϵ such that

$$\|\nabla(u - u^\epsilon)\| \leq C\epsilon \|f\|_{L^2(\Omega)}.$$

Hint: You can use (without proof) the following inequality: There exist a constant c such that for all $v \in H^1_{\#}(\Omega)$ there holds

$$\|v\|_{L^2(\Omega)} \leq c\|\nabla v\|_{L^2(\Omega)}.$$

(2) Let V_h be the finite element space

$$V_h := \{v_h \in C^0(\overline{\Omega}) \; : \; v|_T \in \mathbb{P}^1, \qquad \forall T \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is a subdivision of Ω made of triangles of diameters $h > 0$.

Consider the discrete problem of finding $u_h^\epsilon \in V_h$ such that

$$a_\epsilon(u_h^\epsilon, v_h) = L(v_h), \qquad \forall v_h \in V_h.$$

(i) Show that u_h^ϵ exists and is unique in V_h; (ii) Prove the following error estimate

$$\|\nabla(u^\epsilon - u_h^\epsilon)\|^2_{L^2(\Omega)} + \epsilon\|u^\epsilon - u_h^\epsilon\|^2_{L^2(\Omega)} \leq c\,(h^2 + \epsilon h^4)|u^\epsilon|^2_{H^2(\Omega)},$$

where c is a constant independent of h and ϵ.

Hint: you can use standard interpolation results without proof.

(3) Assume that $|u^\epsilon|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ for a constant C independent of ϵ and derive an error estimate for $\|\nabla(u - u_h^\epsilon)\|_{L^2(\Omega)}$. What is the optimal choice for ϵ?

Problem 2. Let T be the unit triangle in $\mathbb{R}^2$, with vertices $v_1 = (0,0)$, $v_2 = (1,0)$, and $v_3 = (0,1)$ and edges $e_1 = v_1v_2$, $e_2 = v_2v_3$ and $e_3 = v_3v_1$. Let $RT_0 = \{(a + cx, b + cy) : a, b, c \in \mathbb{R}\}$ (so that members of RT_0 are vector functions over T, and $[\mathbb{P}_0]^2 \subsetneq RT_0 \subsetneq [\mathbb{P}_1]^2$). Finally, let $\sigma_i(\vec{u}) = \int_{e_i} \vec{u} \cdot \vec{n}_i$, where $\vec{n}_i$ is the outward pointing unit normal vector to T on e_i, and let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$.

(a) Show that (T, RT_0, Σ) is unisolvent.

(b) Find a basis $\{\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3\}$ for RT_0 that is dual to Σ, i.e. $\sigma_i(\vec{\varphi}_j) = \delta_{ij}$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

(c) Let $(\Pi\vec{u})(x) = \sum_{i=1}^3 \sigma_i(\vec{u})\vec{\varphi}_i(x)$, $x \in T$ and $\vec{u} \in [H^1(T)]^2$. Show that

$$\|\vec{u} - \Pi\vec{u}\|_{[L_2(T)]^2} \leq C|u|_{[H^1(T)]^2}, \; u \in [H^1(T)]^2.$$

Note: You may use standard analysis results such as trace and Poincaré inequalities without proof, but specify carefully which inequalities you are using.

Problem 3. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $T > 0$ be a given final time and $\mathbf{b}$ be a given smooth vector valued function satisfying

$$\operatorname{div}(\mathbf{b}(x,t)) = 0 \quad (x,t) \in \Omega \times [0,T] \quad \text{and} \quad \mathbf{b}(x,t) = 0 \quad (x,t) \in \partial\Omega \times [0,T].$$

Consider the time-dependent problem

$$\frac{\partial u}{\partial t}(x,t) + \mathbf{b}(x,t) \cdot \nabla u(x,t) = 0, \qquad (x,t) \in \Omega \times (0,T)$$

together with the initial condition $u(x,0) = u_0$, $x \in \Omega$.

Let $\mathcal{T}_h$ be a subdivision of Ω made of triangles and

$$V_h := \{v_h \in C^0(\overline{\Omega}) \ : \ v_h|_K \in \mathbb{P}_1 \ \forall K \in \mathcal{T}\}.$$

Choose an integer $N \geq 2$, set $k := T/N$ and $t_n := nk$. Let $u_h^0 \in V_h$ be a given approximation of u_0. For $1 \leq n \leq N$ define $u_h^n \in V_h$ recursively by the relations

$$\frac{1}{k}\int_\Omega (u_h^n(x) - u_h^{n-1}(x))v_h(x)\,dx + \int_\Omega (\mathbf{b}(x,t_n) \cdot \nabla u_h^n(x))v_h(x)\,dx = 0, \qquad \forall v_h \in V_h.$$

- Prove that given $u_h^n \in V_h$, the above finite dimensional system has a unique solution $u_h^{n+1} \in V_h$.
- Prove that for $1 \leq n \leq N$
$$\|u_h^n\|_{L^2(\Omega)} \leq \|u_h^0\|_{L^2(\Omega)}.$$
- Is the matrix representing the finite dimensional system symmetric ? Justify your answer.