Applied/Numerical Analysis Qualifying Exam

January 14, 2015

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
January 14, 2015

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. State and prove the Shannon sampling theorem.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = (x\phi(x))'$ for some $\phi \in \mathcal{D}$ if and only if $\int_{-\infty}^{\infty} \psi(x)dx = \int_{0}^{\infty} \psi(x)dx = 0$.
(d) Use 2(c) to show that if $T \in \mathcal{D}'$ satisfies $xT'(x) = 0$, then $T(x) = c_1 1 + c_2 H(x)$, where H is the Heaviside step function.

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the Banach space of bounded operators on $\mathcal{H}$. In addition, let $\mathcal{C}(\mathcal{H})$ be the subspace of $\mathcal{B}(\mathcal{H})$ comprising the compact operators on $\mathcal{H}$.

(a) Show that $\mathcal{C}(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$.
(b) Define the term *Hilbert-Schmidt operator*. Show that if K is Hilbert-Schmidt, then it is compact. (You may assume that $K \in \mathcal{B}(\mathcal{H})$.)
(c) Find the Green's function $G(x, y)$ for boundary value problem

$$u'' = f,\ u(0) = 0,\ u(1) = 0, \tag{3.1}$$

and show that $Gf(x) = \int_0^1 G(x, y)f(y)dy$ is compact.

Problem 4. Let $\mathcal{P}$ be the set of all polynomials in x.

(a) State and sketch a proof of the Weierstrass approximation theorem.
(b) Show that $\mathcal{P}$ is dense in $L^2[0, 1]$.
(c) Let $\mathcal{U} := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that $\mathcal{U}$ is a complete set in $L^2[0, 1]$.

Applied/Numerical Analysis Qualifying Exam

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Cover Sheet – Numerical Analysis Part

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NUMERICAL ANALYSIS QUALIFIER

January, 2015

Problem 1. Let f be a given function in $L^2(0,2)$ and $u \in H^1(0,2)$ be the solution of

$$a(u,v) := \int_0^2 (u'v' + x\,uv)dx = \int_0^2 fv dx =: L(v), \qquad \text{for all } \; v \in H^1(0,2). \tag{1.1}$$

(a) Derive the strong form of problem (1.1) assuming that the solution u is smooth.
(b) Show that the bilinear form $a(\cdot,\cdot)$ is coercive in $H^1(0,2)$. You first prove that for $v \in H^1(0,2)$

$$\|v\|^2_{L^2(0,2)} \le C \left(\int_1^2 v^2(x)dx + \|v'\|^2_{L^2(0,2)} \right) \tag{1.2}$$

with some positive constant C and then show that the left hand side of this inequality is bounded by $a(v,v)$.
(c) Let $\mathcal{T}_h$, $0 < h < 1$, be quasi-uniform partition of $(0,2)$. The elements of this partitioning will be denoted by τ. Set

$$V_h := \{v_h \in H^1(0,2) : \; v_h|_\tau \in \mathbb{P}^1, \quad \tau \in \mathcal{T}_h\}$$

where $\mathbb{P}^1$ denotes the space of linear polynomials on $\mathbb{R}^2$ and consider the following Galerkin FEM: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) := \int_0^2 u_h' v_h' + Q_h(x\,u_h v_h) = L(v_h), \quad \forall v_h \in V_h. \tag{1.3}$$

where for computing the integral $\int_0^2 x u_h v_h dx$ we have used the quadrature

$$Q_h(w) := \sum_{\tau \in \mathcal{T}_h} \frac{|\tau|}{2}(w(P_1) + w(P_2)),$$

with P_1 and P_2 being the end points of the subinterval τ.
Using similar argument as in the proof of the inequality (1.2), show that $a_h(\cdot,\cdot)$ is coercive in $H^1(0,2)$, i.e. there is a constant $\alpha_0 > 0$, independent of h, such that

$$a_h(v_h, v_h) \ge \alpha_0 \|v_h\|^2_{H^1(0,2)} \quad \forall v_h \in V_h.$$

Problem 2. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, and let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω with element diameters uniformly equivalent to h. Let also $V_h \subset H_0^1(\Omega)$ be a piecewise linear Lagrange finite element space. You may assume the existence of an interpolation operator $I_h : H_0^1(\Omega) \to V_h$ satisfying

$$\|u - I_h u\|_{L_2(\Omega)} + h\|u - I_h u\|_{H^1(\Omega)} \le Ch^2 |u|_{H^2(\Omega)}.$$

(a) Let $u(t) \in H_0^1(\Omega)$ $(0 \le t \le T)$, u_0, and f be sufficiently smooth such that

$$\int_\Omega u_t v\; dx + \int_\Omega \nabla u \cdot \nabla v\; dx = \int_\Omega fv\; dx, \; v \in H_0^1(\Omega), \; 0 < t \le T,$$
$$u(x,0) = u_0(x), \; x \in \Omega.$$

Write down the spatially semidiscrete (i.e., discretized in space but not in time) finite element formulation of this problem. Denote by u_h the solution to these finite element equations.

(b) For $0 < t \le T$, let now $\tilde{u}_h(t)$ be the *elliptic* finite element approximation to $u(t)$. That is

$$\int_\Omega \nabla \tilde{u}_h(t) \cdot \nabla v_h \, dx = \int_\Omega \nabla u(t) \cdot \nabla v_h \, dx, \ v_h \in V_h.$$

Prove that

$$\int_\Omega (u_h - \tilde{u}_h)_t v_h \, dx + \int_\Omega \nabla (u_h - \tilde{u}_h) \cdot \nabla v_h \, dx = \int_\Omega (u - \tilde{u}_h)_t v_h \, dx, \ v_h \in V_h, \ 0 < t \le T.$$

(c) Next recall Gronwall's Lemma, which states that if σ and ρ are continuous real functions with $\sigma \ge 0$ and $c \ge 0$ is a constant, and if

$$\sigma(t) \le \rho(t) + c \int_0^t \sigma(s) \, ds, \ t \in [0, T],$$

then

$$\sigma(t) \le e^{ct} \rho(t), \ t \in [0, T].$$

Using this result, prove that

$$\|(u_h - \tilde{u}_h)(T)\|^2_{L_2(\Omega)} \le C(T) \left(\|(u_h - \tilde{u}_h)(0)\|^2_{L_2(\Omega)} + \int_0^T \|(u - \tilde{u}_h)_t(s)\|^2_{L_2(\Omega)} \, ds \right).$$

(d) For the final part you will need the following intermediate result. Given $v \in H^1_0(\Omega) \cap H^2(\Omega)$, let $v_h \in V_h$ satisfy

$$\int_\Omega \nabla v_h \cdot \nabla w_h dx = \int_\Omega \nabla v \cdot \nabla w_h dx, \quad \text{all } w_h \in V_h.$$

Then

$$\|v - v_h\|_{L_2(\Omega)} \le C h^2 |v|_{H^2(\Omega)}.$$

Assuming this result and additionally that $\|(u - u_h)(0)\|_{L_2(\Omega)} \le C h^2 |u(0)|_{H^2(\Omega)}$, prove that

$$\|(u - u_h)(T)\|_{L_2(\Omega)} \le C(T) h^2 \left(|u(0)|_{H^2(\Omega)} + \left(\int_0^T |u_t|^2_{H^2(\Omega)} \right)^{1/2} \right).$$

Problem 3. Let $(\tau, \mathbb{P}^3, \Sigma)$ be a finite element, where τ is a triangle with vertexes P_1, P_2, and P_3, $\mathbb{P}^3$ is the set of **cubic polynomials**, and Σ is the set of degrees of freedom that consists of the values at 9 points on the boundary, namely, three vertexes and two additional points on each side of τ, chosen in such way that together with the vertexes the four points are equally spaced on the side. The tenth degree of freedom is the average of the function over the triangle normalized by the area of the triangle. This means that the "nodal" basis function ϕ_{10} corresponding to this "degree of fredom" should be zero on the boundary of τ and should satisfy $\frac{1}{|\tau|} \int_\tau \phi_{10} dx = 1$. Find ϕ_{10} and the nodal basis functions of this element corresponding to the vertex P_1.
Hint: Use homogeneous (area) coordinates λ_1, λ_2, and λ_3.