APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFYING EXAMINATION
JANUARY 2009

Policy on misprints. *The qualifying examination committee tries to proofread the examinations as carefully as possible. Nevertheless, there may be a few misprints. If you are convinced that a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.*

1. Part I: Applied Analysis

Work 3 out of 4 problems of this part.

Problem 1. Consider the equation $Lu = f,\ \lambda_1(u) = 0,\ \lambda_2(u) = 0$, where L is a second order linear differential operator. A Green's function $g(x, y)$ for L must satisfy $\lambda_1(g(x, y)) = 0, \lambda_2(g(x, y)) = 0$, where y is fixed, λ_1 and λ_2 are liner functionals, and g is considered as a function of x.

(a) List the other properties $g(x, y)$ must satisfy.

(b) Consider the equation $u''(x) = f(x),\ u(0) = 0,\ \int_0^1 u(t)dt = 0$. Find the Green's function for this equation. (Hint: the Green's function has the form $u_1(\cdot)u_2(\cdot)$ where u_1 is a solution to $u''(x) = 0,\ u(0) = 0$ while u_2 is a solution to $u''(x) = 0$.

(c) Write down a solution to $u''(x) = f(x),\ u(0) = 0,\ \int_0^1 u(t)dt = 0$.

Problem 2. (a) State the Courant Minimax Principle.

(b) Prove an inequality relating the eigenvalues of a symmetric matrix before and after one of its diagonal elements is increased.

(c) Use this inequality and the minimax principle to show that the smallest eigenvalue of the following matrix is negative:

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 3 \end{pmatrix}$$

Problem 3. Let K be a compact, self-adjoint operator on a Hilbert space H and suppose $(I - \lambda K)$ is bounded below, i.e., $\inf_{||u||=1} ||(I - \lambda K)u|| > 0$.

(a) Explain why $(I - \lambda K)u = f$ can always be solved whenever $f \in H$.

(b) Explain how to solve $(I - \lambda K)u = f$ explicitly in terms of the eigenfunctions of K.

Problem 4. (a) Prove the following theorem: If $\{P_n\}$ is a sequence of projections with the property that $||P_n u - u|| \to 0$ as $n \to \infty$ for every $u \in H$, and if $(I - \lambda K)^{-1}$ exists, then u_n, the solution of $(I - \lambda K)u_n = P_n f$, converges to the solution of $(I - \lambda K)u = f$ as $n \to \infty$.

(b) Apply this theorem to sketch a way to find an approximate solution of the integral equation

$$u(x) + \int_0^1 k(x, y)u(y)dy = f(x)$$

using piecewise linear finite elements. For simplicity assume $k(x, y)$ and $f(x)$ are continuous functions of their arguments and define $\phi_k(x)$ to be the piecewise linear continuous functions with $\phi_k(x_j) = \delta_{k,j}$ and linear on all the intervals $[x_j, x_{j+1}]$ where $x_j = j/n$. Also assume that $\{P_n\}$ are interpolating projections.

2. Part II: Numerical Analysis

Work 2 out of 3 problems of this part.

Problem 1. Let $\Omega = (0, 1)$ and u be the solution of the boundary value problem

$$\begin{aligned} u^{(4)} - (k(x)u')' + q(x)u &= f(x), \qquad \text{for } \ x \in \Omega, \\ u(0) = u''(0) &= 0, \\ u(1) = 0, \ u''(1) + \beta u'(1) &= \gamma, \end{aligned}$$

where $k(x) \geq 0$, $q(x) \geq 0$, $f(x)$, γ, and $\beta > 0$ are given data.

(a) Derived the weak formulation of this problem. Specify the appropriate Sobolev spaces and show that the corresponding bilinear form is coercive.

(b) Suggest a finite element approximation to this problem using piece-wise polynomial functions over a uniform partition of Ω into subintervals with length h.

(c) Derive an error estimate for the FE solution.

Problem 2. Let $\Omega = (0, 1)^2$ and u be the solution of the second order elliptic problem:

$$\begin{aligned} -\Delta u := -u_{x_1x_1} - u_{x_2x_2} &= f(x), \qquad \text{for } \ x \in \Omega, \\ \frac{\partial u}{\partial n} + u &= g(x), \qquad \text{for } \ x \in \partial\Omega, \end{aligned}$$

where n is the outer unit normal vector to the boundary $\partial\Omega$ and $f(x)$ and $g(x)$ are given functions.

(a) Derived the weak formulation of this problem in the form $a(u,v) = F(v)$, where $a(u,v)$ and $F(v)$ are appropriate linear and bilinear form defined on the Sobolev space $H^1(\Omega)$.

(b) Let S_h be a finite element space of continuous piece-wise polynomial functions defined over a regular partitioning of Ω into triangles ane let $a_h(u,v)$ and $F_h(v)$ be the bilinear forms where all integrals are comptuited approximately. Derive Strang's lemma for the error of the FEM: find $u_h \in S_h$ such that $a_h(u_h, v) = F_h(v), \ \forall \ v \in S_h$.

(c) Let S_h be the finite element space of piece-wise linear functions. Let all integrals in $a(u,v)$ and $F(v)$ be comptuited using quadratures. Namely, for τ and e being triangle and edge defined by the vertexes P_1, P_2, P_3 and P_1, P_2, respectively,

$$\int_\tau w(x)dx \approx \frac{|\tau|}{3}(w(P_1) + w(P_2) + w(P_3)), \quad \int_e w(x)ds \approx \frac{|e|}{2}(w(\alpha) + w(\beta)),$$

where $|\tau|$ is the area of τ and $|e|$ is the length of e, and α and β are the Gaussian quadrature nodes. Explain why $a(w,v) = a_h(w,v)$ for all $w, v \in S_h$.

(d) Using the estimate of Part (b) estimate the error $\|u - u_h\|_{H^1}$.

Problem 3. Let $\Omega = (0,1)^2$ and u be the solution of the elliptic problem:

$$-\Delta u + u = f(x), \qquad \text{for } \ x \in \Omega, \quad u = g(x), \qquad \text{for } \ x \in \partial\Omega.$$

(a) Let $\omega_h = \{x = (x_{1,i}, x_{2,j}) : \ x_{1,i} = ih, x_{2,j} = jh, \ i,j = 0,1,\ldots,N, h = 1/N\}$ be a square mesh in Ω. Write down the 5-point stencil finite difference scheme for the approximate solution $U_{ij} = U(x_{1,i}, x_{2,j})$ of the above problem. Estimate the local truncation error.

(b) Show that: $\displaystyle \max_{x \in \omega_h} |U(x)| \le \max_{x \in \omega_h \cap \partial\Omega} |g(x)| + \max_{x \in \omega_h} |f(x)|.$

(c) Using this a priori estimate and the estimation of the local truncation error in (a) conclude that for sufficiently smooth solution $u(x)$ the following error estimate (with a constant independent of h):

$$\max_{x \in \omega_h} |U(x) - u(x)| \le Ch^2.$$