**Complex analysis qualifying exam, August 2014.**

**1.** Give the statements of

(a) Runge's Theorem;
(b) Schwarz' Lemma.

**2.** a) Find and classify all isolated singularities of

$$f(z) = \frac{z^2(z-\pi)}{\sin^2 z} \quad \text{and} \quad g(z) = (z^2-1)\cos\frac{1}{z-1}.$$

b) Find the residue of $f$ at $z = 2\pi$ and the residue of $g$ at $z = 1$.

**3.** Let $u$ be a bounded harmonic function in the first quadrant $\Omega = \{\Re z > 0,\ \Im z > 0\}$. Suppose that the limit

$$\lim_{z\to\xi,\ z\in\Omega} u(z)$$

is equal to 1 for all $\xi \in (0,1)$ and is equal to 0 for all $\xi \in \partial\Omega \setminus [0,1]$. Find $u\left(\frac{1+i}{\sqrt{2}}\right)$.

**4.** Let $f$ be an entire function. Suppose that $f$ satisfies

$$|f(x+iy)| \le \frac{1}{|y|}$$

for all $x, y \in \mathbb{R}$. Prove that $f$ is identically zero.

**5.** Find a formula for a conformal map from $D_1$ to $D_2$, where $D_1$ is the unit disk with a slit, $D_1 = \{|z| < 1,\ z \notin [1/2, 1]\}$, and $D_2$ is the strip $D_2 = \{|\Re z| < 1\}$.

**6.** Prove that if $0 < |z| < 1$, then $\frac{1}{4}|z| < |1 - e^z| < \frac{7}{4}|z|$.

**7.** Prove that the equation

$$az^3 - z + b = e^{-z}(z+2)$$

has two solutions in the right half-plane $\{\Re z > 0\}$ when $a > 0$ and $b > 2$.

**8.** Let $f$ be a bounded analytic function in the upper half-plane $\mathbb{C}_+$. Suppose that

$$f(in) = e^{-n}$$

for all $n \in \mathbb{N}$. Find $f(1+i)$. (You need to explain why the value that you found is the only possible.)

**9.** Let $f_n : \mathbb{D} \to \mathbb{D}$ be a sequence of holomorphic functions in the unit disk $\mathbb{D}$. Suppose that $f_n(z) \to 1$ for some $z \in \mathbb{D}$. Prove that then $f_n$ converges to 1 normally in $\mathbb{D}$.

**10.** Suppose that $f$ is an entire function, and $g$ is a holomorphic function in the punctured disk $\{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$. If the composite function $f \circ g$ has a simple pole at the origin, then what can you deduce about the functions $f$ and $g$?