Complex Analysis Qualifying Exam, January 2023

$\mathbb{D}$ denotes the unit disc $B(0,1)$.

Problem 1: Let f be analytic in $\mathbb{D}$. Show that if $\sum_{n=0}^{\infty} |f^{(n)}(0)| < \infty$, then f is (the restriction of) an entire function.

Problem 2: Compute

$$\int_0^\infty \frac{x}{1+x^5}\,dx\ .$$

Problem 3: Suppose the open set $\Omega \subseteq \mathbb{C}$ contains $\overline{\mathbb{D}}$. Assume f is analytic in Ω, with n simple zeros in $\mathbb{D}$, and no zeros on $\partial\mathbb{D}$. Show: $Re(f)$ has at least $2n$ zeros on $\partial\mathbb{D}$.

Problem 4: Find all Möbius transformations that map $\mathbb{D}$ to itself and map the circle $\{z : |z - 2/5| = 2/5\}$ to a circle centered at the origin. Prove that you have found them all.

Problem 5: Let $f : \mathbb{D} \to \mathbb{D}$ be analytic, with $f(a) = a$ and $f(b) = b$ for two (distinct) points a and b in $\mathbb{D}$. Show that $f(z) = z$ for all $z \in \mathbb{D}$.

Problem 6: Let f be analytic in the upper half plane and continuous on its closure. Assume that f satisfies the estimate $|f(z)| \le M|z|^{-r}$, $z \ne 0$, for strictly positive constants M and r. Show that if $Im(z) > 0$, then

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt\ .$$

Problem 7: Let $p(z)$ and $q(z)$ be two polynomials, with $deg(p) = n \ge 1$. Define Ω to be the set of those points $z \in \mathbb{C}$ where the set $p^{-1}(z) := \{w : p(w) = z\}$ consists of n (distinct) points.

a) Show: Ω is open.

b) Show that $\widetilde{q}(z) := q(w_1) + q(w_2) + \cdots + q(w_n)$ is analytic in Ω, where $\{w_1, w_2, \cdots, w_n\} = p^{-1}(z)$.

Problem 8: Let f be a meromorphic function in a neighborhood of $\overline{\mathbb{D}}$, with no pole on $\partial\mathbb{D}$. Prove that if $f(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$, then f is a rational function.

Problem 9: Prove that there is no nonconstant harmonic function $u : \mathbb{C} \to \mathbb{R}$ such that $u(z) \le 0$ for all $z \in \mathbb{C}$.

Problem 10: Since $\cos(z)$ is even, $\cos(\sqrt{z})$ is an entire function. Determine both the order and the genus of this function.