Real Analysis Qualifying Exam; January, 2013.

Work as many of these ten problems as you can in four hours. Start each problem on a new sheet of paper.

#1. Let f be a Lebesgue integrable, real–valued function on $(0,1)$ and for $x \in (0,1)$ define

$$g(x) = \int_x^1 t^{-1} f(t)\, dt.$$

Show that g is Lebesgue integrable on $(0,1)$ and that $\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx$.

#2. Let $f_n \in C[0,1]$. Show that $f_n \to 0$ weakly if and only if the sequence $(\|f_n\|)_{n=1}^{\infty}$ is bounded and f_n converges pointwise to 0.

#3. Let (X,μ) be a measure space with $0 < \mu(X) \leq 1$ and let $f : X \to \mathbf{R}$ be measurable. State the definition of $\|f\|_p$ for $p \in [1,\infty]$. Show that $\|f\|_p$ is a monotone increasing function of $p \in [1,\infty)$ and that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

#4. (a) Is there a signed Borel measure μ on $[0,1]$ such that

$$p'(0) = \int_0^1 p(x)\, d\mu(x)$$

for all real polynomials p of degree at most 19?

(b) Is there a signed Borel measure μ on $[0,1]$ such that

$$p'(0) = \int_0^1 p(x)\, d\mu(x)$$

for all real polynomials p?

(Justify your answers).

#5. Let $\mathcal{F}$ be the set of all real–valued functions on $[0,1]$ of the form

$$f(t) = \frac{1}{\prod_{j=1}^{n}(t - c_j)}$$

for natural numbers n and for real numbers $c_j \notin [0,1]$. Prove or disprove: for all continuous, real–valued functions g and h on $[0,1]$ such that $g(t) < h(t)$ for all $t \in [0,1]$, there is a function $a \in \operatorname{span} \mathcal{F}$ such that $g(t) < a(t) < h(t)$ for all $t \in [0,1]$.

#6. Let $k : [0,1] \times [0,1] \to \mathbf{R}$ be continuous and let $1 < p < \infty$. For $f \in L^p[0,1]$, let Tf be the function on $[0,1]$ defined by

$$(Tf)(x) = \int_0^1 k(x,y) f(y)\, dy.$$

Show that Tf is a continuous function on $[0,1]$ and that the image under T of the unit ball in $L^p[0,1]$ has compact closure in $C[0,1]$.

#7. (a) Define the *total variation* of a function $f : [0,1] \to \mathbf{R}$ and *absolute continuity* of f.

(b) Suppose $f : [0,1] \to \mathbf{R}$ is absolutely continuous and define $g \in C[0,1]$ by

$$g(x) = \int_0^1 f(xy)\, dy.$$

Show that g is absolutely continuous.

#8. (a) State the definition of absolute continuity, $\nu \ll \mu$, for positive measures μ and ν, and state the Radon–Nikodym Theorem, (or the Lebesgue–Radon–Nikodym Theorem, if you prefer.)

(b) Suppose that we have $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ for positive measures ν_i and μ_i on measurable spaces $(X_i, \mathcal{M}_i)$, $(i = 1, 2)$. Show that we have $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$, and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x, y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y).$$

#9. (a) Let E be a nonzero Banach space and show that for every $x \in E$ there is $\phi \in E^*$ such that $\|\phi\| = 1$ and $|\phi(x)| = \|x\|$.

(b) Let E and F be Banach spaces, let $\pi : E \to F$ be a bounded linear map and let $\pi^* : F^* \to E^*$ be the induced map on dual spaces. Show that $\|\pi^*\| = \|\pi\|$.

#10. Let X be a real Banach space and suppose C is a closed subset of X such that
(i) $x_1 + x_2 \in C$ for all $x_1,\, x_2 \in C$,
(ii) $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$,
(iii) for all $x \in X$ there exist $x_1,\, x_2 \in C$ such that $x = x_1 - x_2$.

Prove that, for some $M > 0$, the unit ball of X is contained in the closure of

$$\{x_1 - x_2 \mid x_i \in C,\ \|x_i\| \le M,\ (i = 1, 2)\}.$$

Deduce that every $x \in X$ can be written $x = x_1 - x_2$, with $x_i \in C$ and $\|x_i\| \le 2M\|x\|$, $(i = 1, 2)$.