Real Analysis Qualifying Exam

January 2021

Each problem is worth 10 points. Work each problem on a separate piece of paper.

1. Let (X, μ) be a finite measure space and $f : X \to [0, \infty)$ an integrable function. For each n set $g_n(x) = f(x)^{1/n}$ for all $x \in X$. Show that the sequence $\{g_n\}$ converges in $L^1(\mu)$ and determine the limit.

2. Let μ be Lebesgue measure on $[0, 1]$ and let A be a closed subset of $[0, 1]$. Prove that $\mu(A) = 0$ if and only if there is a sequence $\{p_n\}$ of polynomials such that

(i) $p_n(x) \geq 0$ for all n and $x \in [0, 1]$,
(ii) $\int_0^1 p_n \, d\mu \to 0$ as $n \to \infty$, and
(iii) $p_n(x) \to \infty$ for all $x \in A$.

3. (a) Let X be a normed space and $\{x_n\}$ a sequence in X. For each n set $y_n = (x_1 + \cdots + x_n)/n$. Show that if $\{x_n\}$ converges then so does $\{y_n\}$.

(b) Consider $[0, 1]$ with Lebesgue measure μ. Show that there exists a sequence $\{f_n\}$ of nonnegative integrable functions on $[0, 1]$ such that f_n converges in measure to zero but the averages $g_n = (f_1 + \cdots + f_n)/n$ do not.

4. Prove or disprove: ℓ_1 and c_0 are isomorphic. (Recall that the Banach space c_0 is the space of sequences $\mathbb{N} \to \mathbb{C}$ which converge to zero, with pointwise vector space operations and supremum norm, and that an isomorphism between Banach spaces is an invertible bounded linear map with bounded inverse.)

5. Let (X, d) be a compact metric space and regard $C(X)^*$ as the space of finite Borel signed measures on X. Let $\{\mu_n\}$ be a weak* convergent sequence of Borel probability measures on X. Recall that the support of a measure on X is the complement of the union of all open sets with zero measure. Show that if the diameter of the support of μ_n tends to zero as $n \to \infty$ then the limit of $\{\mu_n\}$ is a point mass. Also, show that the converse is false.

6. A sequence $\{x_n\}$ in a normed space X is said to be weakly Cauchy provided that for each $x^* \in X^*$ the sequence $\{x^*(x_n)\}$ is a convergent sequence of scalars.

(a) Prove that a weakly Cauchy sequence in a normed space is norm bounded.

(b) Prove that a weakly Cauchy sequence in a reflexive Banach space is weakly convergent.

7. Let K be a nonempty closed convex subset of $L_2(0, 1)$. Prove or disprove that there must exist an x in K such that $\|x\| = \inf_{y \in K} \|y\|$.

8. Prove that if X is a separable Banach space then there is a bounded linear operator $T : \ell_2 \to X$ such that $T\ell_2$ is dense in X.

9. Let (X, μ) be a finite measure space and let $\{A_n\}$ be a sequence of measurable subsets of X whose indicator functions χ_{A_n} converge in $L^1(\mu)$. Show that the limit is a.e. equal to the indicator function of some measurable set.

10. Consider $[0, 1]$ with Lebesgue measure μ. For each n define $f_n = \sum_{k=0}^{2^n-1} (-1)^k \chi_{A_k}$ where $A_k = [k/2^n, (k + 1)/2^n]$. Show that $f_n \to 0$ weakly in $L^1[0, 1]$.