Real Analysis Qualifying Exam
January 10, 2022

There are ten equally weighted problems. Solve as many of them as you can in the alotted time. Refer by name to any theorems that you use.

1. Prove or disprove that if $(f_n)_{n=1}^{\infty}$ is a sequence of Lebesgue integrable functions

$$f_n : [0,1] \to \mathbb{R} \quad \text{such that} \quad \lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{R})} = 0,$$

then for at least one value $x \in [0,1]$, we have

$$\lim_{n\to\infty} f_n(x) = 0.$$

2. Let $\mathcal{A}$ be the set of all real valued functions on $[0,1]$ for which $f(0) = 0$ and

$$|f(t) - f(s)|^4 \le t - s \quad \text{for all} \quad 0 \le s < t \le 1.$$

Prove that $\mathcal{A}$ is a compact subset of $L^2[0,1]$. Don't forget to justify that $\mathcal{A}$ is closed in $L^2[0,1]$.

3. Let E be a subset of $\mathbb{R}$ that has positive Lebesgue measure and set

$$S = \{f \in L^1(\mathbb{R}) : 1_E f = 0 \text{ a.e.}\}.$$

Prove or disprove that S is closed in $L^1(\mathbb{R})$.

4. Prove or disprove that there is a sequence of real polynomials $(p_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \int_0^1 |p_n(t)|\, dt = 1,$$

but such that for all $t \in [0,1]$, $\lim_{n\to\infty} p_n(t) = 0$.

5. (a, for 4 points) Show (directly from the relevant definitions) that every separable metric space is a 2nd countable topological space.
(b, for 4 points) Supppose μ is a Borel measure on a 2nd countable topological space X. Show that there exists a largest subset U of X that is both open and μ-null.
(c, for 2 points) Give an example of a topological space X and a Borel measure μ on X for which there is no largest subset of X that is both open and μ-null.

6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and suppose $f \in L^p(\mu)$ for some $p \in (0,\infty)$. Show

$$\lim_{n\to\infty} \int |f|^{1/n}\, d\mu = \mu\big(\{x \in X : f(x) \neq 0\}\big).$$

7. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence in a Banach space X that converges weakly to $x \in X$. Show that

$$\liminf_{n\to\infty} \|x_n\| \ge \|x\|.$$

8. Let $1 \leq p \leq \infty$ and let $f \in L^p(\mathbb{R})$. Show that

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+t^2}\, dt < \infty.$$

9. Show that for every positive integer n, there is a regular, Borel, signed measure μ on $[0, 1]$ such that for all real polynomials P of degree $\leq n$,

$$P'(1/2) = \int P(x)\, d\mu(x).$$

where P' is the derivative of P.

10. Let X be the vector space of all real polynomials in one variable. Prove or disprove that there exists a norm on X making X into a Banach space.