Random geometry and Percolation

- Percolation theory on $\mathbb{Z}^d$ and general graphs.
- Critical behavior
- Modern tools such as renormalization, stochastic dimension.
- Random walk on random graphs.
- Correlated percolation models: Random interlacements and Gaussian free field.

<u>Books:</u>

- Grimmett: percolation 2nd edition.

online:

- Benjamini – Coarse Geometry and Randomness
- Drewitz, Rath, Sapozhnikov – An introduction to Random Interlacements
- papers.

Credit: - Exercises (every two weeks).

- Can choose to present a one hour lecture (not easy).

Percolation on graphs:

Given a graph $G = (V, E)$

Percolation on G is a random diluting of the graph.

Let $\Omega = \{0,1\}^E$, $\mathcal{F}$ - cylinder σ-algebra (events which depend on a finite # of edges), $p \in [0,1]$, $\mu_p(1) = p$, $\mu_p(0) = 1-p$

$\mathbb{P}_p = \prod_{e \in E} \mu_p$ (product measure). Bond percolation on G_x is $(\Omega, \mathcal{F}, \mathbb{P}_p)$ with parameter p

can be realized by random variables. π_e - projection map to edge e.

Then $\{\pi_e\}_{e \in E}$ are i.i.d $\sim \mu_p$.

Percolation is the simplest model which exhibits critical phenomena: there is a natural parameter in the model which admits a drastic behavior change.

What are the typical questions:

<u>Definition!</u> The component of a vertex $v \in V$ is the set of all vertices that can be reached from v using a path of open edges ($\pi_e = 1$).

$$C(v) = \{ w \in V : \exists n \in \mathbb{N},\ v_0, v_1, \ldots, v_n \in V,\ v_0 = v,\ v_n = w,\ (v_i, v_{i+1}) \in E$$
$$\pi_{v_i, v_{i+1}} = 1 \ \forall\, 0 \le i \le n-1 \},$$

Q_1: Are there infinite components?

Q_2: Does this depend on p?

Q_3: Can one compute the critical value? (still open in $\mathbb{Z}^3$).

Q_4: How many infinite components are there?

Q_5: What is the size of finite open components?

Q_6: What is the typical graph distance between two vertices in an open component.

Two main examples to keep in mind $G = \mathbb{Z}^d$ or $\mathbb{T}_d$ (d-regular tree).

Theorem: Let $G = \mathbb{Z}^d$, then

1. if $d > 1$ and $p > 3/4$ then 0 is connected to ∞ with positive probability.

2. if $d \geq 2$ and $p < \frac{1}{3}$ then 0 is not connected to ∞ w.p 1.

Remark: 1. enough to prove first part for $d = 2$. (subgraph).

2. Theorem suggests existence of a phase transition for existence of inf cluster.

<u>Definition:</u> for a graph G and a fixed root ρ,

$$p_c(G,\rho) = \inf\{0 \le p \le 1 : \mathbb{P}_p(\rho \leftrightarrow \infty) > 0\}.$$

Ex. If G is a connected graph then $p_c(G,\rho)$ does not depend on ρ.

<u>Corollary:</u> $\frac{1}{3} < p_c(\mathbb{Z}^2) < 3/4$ (non trivial).

<u>Ex!</u> Is there a connected graph G with $p_c(G) = 1$? ($\mathbb{Z}$)

what about $\mathbb{Z} \times \mathbb{Z}_2$ (define Cartesian product).

Thm: Let G be a connected graph with $\deg(v) \leq d$ $\forall v \in V$ then

$$p_c(G) \geq \frac{1}{d-1}$$

Proof: If $\rho \leftrightarrow \infty$ then there is an open simple (self avoiding) path from ρ to ∞. The # of self avoiding paths of length n starting at ρ is at most $d(d-1)^{n-1}$ (tight for what graph?) The prob for such a path to be open is p^n

$$\Rightarrow \mathbb{P}_p(\rho \leftrightarrow \infty) \leq \mathbb{P}_p(\exists \gamma \text{ open, self avoiding}, |\gamma| = n, \text{ from } \rho) =$$

$$= \mathbb{P}\Big(\bigcup_{\substack{\gamma \text{ self avoiding} \\ |\gamma| = n}} \{\gamma \text{ is open}\}\Big) \leq \sum \mathbb{P}(\;) = d(d-1)^{n-1} p^n \xrightarrow[\text{if } p < \frac{1}{d-1}]{n \to \infty} 0$$

∎

Corollary: 1. $p_c(\mathbb{T}_d) \geq \frac{1}{d-1}$ (actually $= \frac{1}{d-1}$)

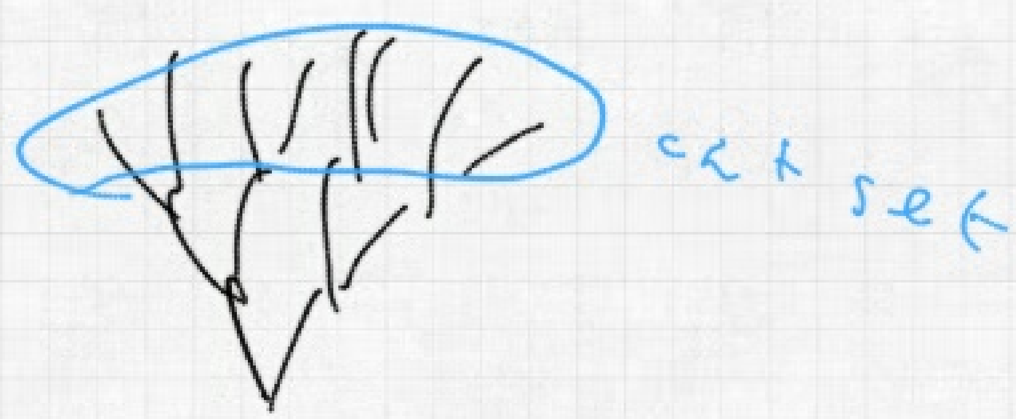
2. $p_c(\mathbb{Z}^d) \geq \frac{1}{2d-1} \Rightarrow p_c(\mathbb{Z}^2) \geq \frac{1}{3}$.

Definition: for G with a root ρ, we say that a set of edges is a cut set if every path from ρ to ∞ must cross an edge from the set.

Example: $G = \mathbb{Z}$ then $\{(-5,-6), (7,8)\}$ is a cut set

$\{(3,4), (100,101)\}$ is not.

$G = \mathbb{T}_2$

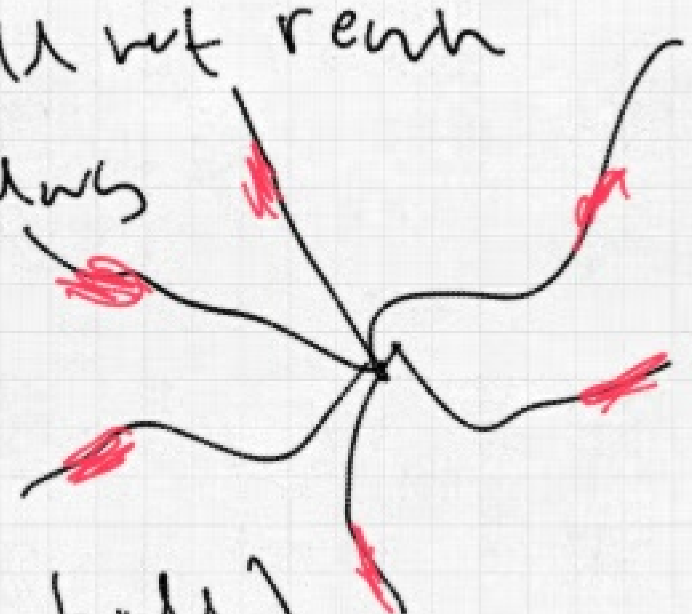


Definition: A cut set is said to be a minimal cut set (MCS) if any $T \subsetneq S$ is not a cut set.

Theorem: Let G be a connected infinite graph with root ρ. If $\exists C > 0$ s.t. $\forall n \geq 0$ $\#\{MCS_S \text{ of size } n\} < C^n$ then $p_c(G) < 1$. (What about $G = \mathbb{Z}$? $n \geq 2$ we get inf)

Proof: If $\rho \not\leftrightarrow \infty$ then there is a MCS which is all closed. If S is a cutset of size n,

(will not reach boundary of the ball)



$$\mathbb{P}_p(S \text{ closed}) = (1-p)^n$$

$$\Rightarrow \mathbb{P}_p(\exists \text{ closed MCS of size } n) \leq (C(1-p))^n$$

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$$\Rightarrow \mathbb{P}_p(\exists \text{ closed mcs}) \le \sum_{n=1}^{\infty} \left(C(1-p)\right)^n \xrightarrow[p\to 1]{} 0$$

$\Rightarrow \exists p_0 < 1 \ \forall p > p_0 \ \ \rho \leftrightarrow \infty$ with positive probability. ∎

Exponential Intersection Tail: (Kesten)

<u>Definition:</u> A rooted graph (G,ρ) admits exponential intersection tail (EIT) if $\exists$ a measure μ on paths in G, supported only on infinite paths from ρ, with the following property: $\exists\, 0<\theta<1$ s.t.

$$\mathbb{P}_{\mu\times\mu}(|\gamma_1 \cap \gamma_2| \ge n) \le \theta^n.$$

Prop: The binary tree has E|T property.

proof: μ uniform over monotone paths (SRW - directed).

$$P_{\mu\times\mu}(|a\cap a'| > n) = \frac{1}{2^n}$$ ☒

Prop: $\mathbb{Z}^2$ does not have E|T.

proof: Let μ be a measure over paths from 0.

$\tilde{\mu}_n(x)$ = prob first hit on $\partial[-n,n]$ is in x.

$$P_{\mu\times\mu}(\text{paths intersect in } \partial[-n,n]^2) \geq \sum_{x\in\partial[-n,n]^2} \tilde{\mu}_n(x)^2 \underset{\text{Cauchy Schwartz}}{\geq} \left(\frac{\sum \mu_n(x)}{\left(\sum_{x\in\partial} 1\right)^{1/2}}\right)^2 = \frac{1}{4n}$$

≥ 1 if $\exists$ intersection.

$$\mathbb{E}_{\mu\times\mu}[\#\text{ of intersections}] = \sum_{n=1}^{\infty} \mathbb{E}_{\mu\times\mu}[\#\text{ of intersections on the shell of size } n] > \sum_{n=1}^{\infty} \frac{c}{n} = \infty$$

but EIT $\Rightarrow$ finite

Open problem: Does loop erased rw on $\mathbb{Z}^d$ admit EIT?

prop: μ uniform on monotone paths admit EIT in $\mathbb{Z}^d$ when $d \geq 4$.

Theorem: If G admits EIT with parameter θ then $p_c(G) \le \theta$.

Proof: Let μ be a measure satisfying EIT-θ.

Measure

$$Z_n = \int_\gamma p^{-n} \mathbb{1}_{\{\gamma[0,n] \text{ is open}\}} \, d\mu$$

without p^{-n} this is μ(paths stay in the open cluster of 0 for n steps).

$\Rightarrow$ need to show $\exists c > 0$ s.t. $\mathbb{P}_p(Z_n > 0) > c$ $\forall n \in \mathbb{N}$.

by Fubini: $\mathbb{E}[Z_n] = \int_\gamma p^{-n} \cdot 1 \cdot p^n \, d\mu(\gamma) = 1$

by Paley Zygmund inequality (Cauchy - Schwartz),

$$\mathbb{P}_p(Z_n > 0) \geq \frac{\mathbb{E}_p[Z_n]^2}{\mathbb{E}_p[Z_n^2]} = 1$$

$\Rightarrow$ we need a uniform upper bound on $\mathbb{P}_p[Z_n^2]$.

$$\mathbb{E}_p[Z_n^2] = \mathbb{E}_p\left[\int_{\gamma_1}\int_{\gamma_2} p^{-2n} \mathbb{1}_{\{\gamma_1 \text{ and } \gamma_2 \text{ are open}\}} d\mu(\gamma_1)\, d\mu(\gamma_2)\right]$$

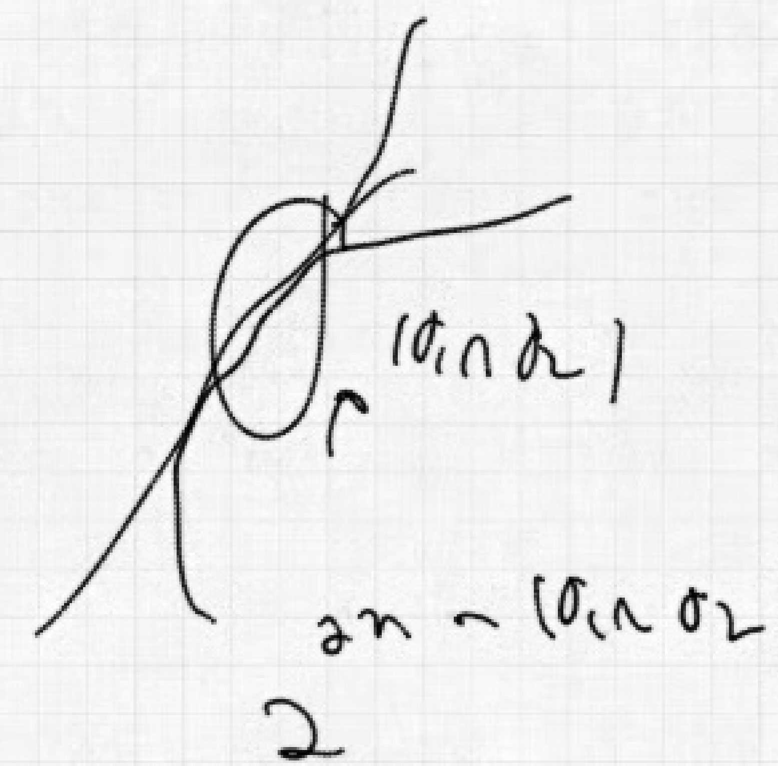
(under "γ_1" and "γ_2": Gil(n) / Gil(n))

$$\leq \int_{\sigma_1}\int_{\sigma_2} \rho^{-|\sigma_1 \cap \sigma_2|} \, d\mu(\sigma_1)\, d\mu(\sigma_2) \leq \sum_{i=d}^{\infty} \left(\frac{\theta}{\rho}\right)^{i}$$

EIT

if $\rho > \theta$ sum is finite ∎

<u>Ex:</u> p_c (d-regular tree) $= \frac{1}{d}$.



<u>Ergodic theory</u>: Given a measure space $(\Omega, \mathcal{F}, \mu)$

and a measurable map $T: \Omega \circlearrowleft$. T is measure preserving if

$\forall A \in \mathcal{F} \quad \mu(A) = \mu(T^{-1}(A))$.

Given $(\Omega, \mathcal{F}, \mu, T)$ $E \in \mathcal{F}$ is called invariant if $T^{-1}(E) = E$.

<u>Definition</u>: A measure preserving space $(\Omega, \mathcal{F}, \mu, T)$ (complete) is called Ergodic if $\forall A \in \mathcal{F}$ s.t. $T^{-1}A = A$, $\mu(A) \in \{0, 1\}$.

<u>Example</u>: percolation on $\mathbb{Z}^d$ $T = \theta_1$ - shift in e_1 direction

<u>Proof</u>: A, B cylinder events $\mu(A \cap T^{-n} B) \to \mu(A)\mu(B)$.

Remark on general events not cylinders

∞

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no strong mixing for inf measures

Definition: A probability preserving transformation $(X, \mathcal{B}, \mu, T)$ is called strong mixing, if for all $E, F \in \mathcal{B}$, $\mu(E \cap T^{-n}F) \xrightarrow[n\to\infty]{} \mu(E)\mu(F)$

Prop: strong mixing $\Rightarrow$ ergodicity.

Proof: If $T^{-1}(E) = E$ then $\mu(E) = \mu(E \cap T^{-n}(E)) \to \mu(E)^2$

$\Rightarrow \mu(E) \in \{0,1\}$.

∎

Prop: $(X, \mathcal{B}, \mu, T)$ is strongly mixing iff $\forall f, g \in L^2$

$$\int f\, g \circ T^n \, d\mu \xrightarrow[n\to\infty]{} \int f\, d\mu \int g\, d\mu \quad \text{equivalently} \quad \mathrm{Cov}(f, g\circ T^n) \xrightarrow[n\to\infty]{} 0.$$

Proof: Ex.

Continue proof (mixing is ergodic): ∀ f, g indicators of cylinders

$\mathrm{Cov}(f, g \circ T^n) \xrightarrow[n\to\infty]{} 0$. Every L^2 function can be approximated in L^2 by a finite linear combination of indicators of cylinders.

s.t. $\|f - f_\varepsilon\|_2 < \varepsilon$, $\|g - g_\varepsilon\|_2 < \varepsilon$.

$$|\mathrm{Cov}(f, g\circ T^n)| \le |\mathrm{Cov}(f - f_\varepsilon, g\circ T^n)| + |\mathrm{Cov}(f_\varepsilon, g_\varepsilon \circ T^n)|$$

$$+ |\mathrm{Cov}(f_\varepsilon, (g - g_\varepsilon)\circ T^n)|$$

$$|\mathrm{Cov}(f, g)| \underset{\text{C-S}}{\le} \|f - \textstyle\int f\|_2 \|g - \int g\|_2 \le (\|f\|_2 + \|f\|_1)(\|g\|_2 + \|g\|_1) \le 2\|f\|_2\, 2\|g\|_2 .$$

$$\le 4\varepsilon\|g\|_2 + o(1) + 4(\|f\|_2 + \varepsilon)\varepsilon \quad \text{as } n \to \infty$$

∎

<u>Theorem:</u> Tempelman: Let $T_1, \dots, T_d$ be θe_i, and

$\{I_r\}_{r \geq 1}$ an increasing sequence of boxes which tends to $\mathbb{Z}^d$.

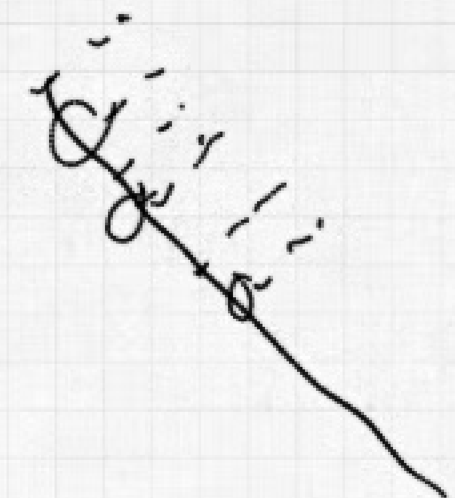
If $f \in L_1$, then

$$\frac{1}{|I_r|} \sum_{\bar{n} \in I_r} f \circ T^{\bar{n}} \xrightarrow[r \to \infty]{} \mathbb{E}\left[f \mid Inv(T_1) \cap \dots \cap Inv(T_d)\right] \text{ a.s.}$$

Uniqueness of the inf cluster:

For $\mathbb{T}_2$ we saw that $p_c = \frac{1}{2}$. Let $\frac{1}{2} < p < 1$

How many inf comp are there?



it's any cluster
edges and
it's open.
(interchanging).

$\mathbb{P}(\infty \text{ inf comp}) = 1$.

Uniqueness in $\mathbb{Z}^d$: (on any amenable transitive graph).

Denote $\Theta_d^{(p)} = \mathbb{P}_p(|C(0)| = \infty)$.

Theorem: If $\theta(p)>0$ then $\mathbb{P}_p(\exists$ a unique inf cluster$)=1$

Proof: $\forall k \in \{0,1,2,\ldots,\infty\}$ let $E_k=\{$# of inf clusters is $k\}$.

E_k is inv event $\Rightarrow$ $\mathbb{P}_p(E_k)\in\{0,1\}$ $\forall k$. Since E_k are

disjoint and cover the space $\exists k$ s.t. $\mathbb{P}_p(E_k)=1$,

note $\mathbb{P}_p(\exists$ inf cluster$)=1$.

Lemma: $\forall k\neq 1$ finite $\mathbb{P}_p(E_k)=0$.

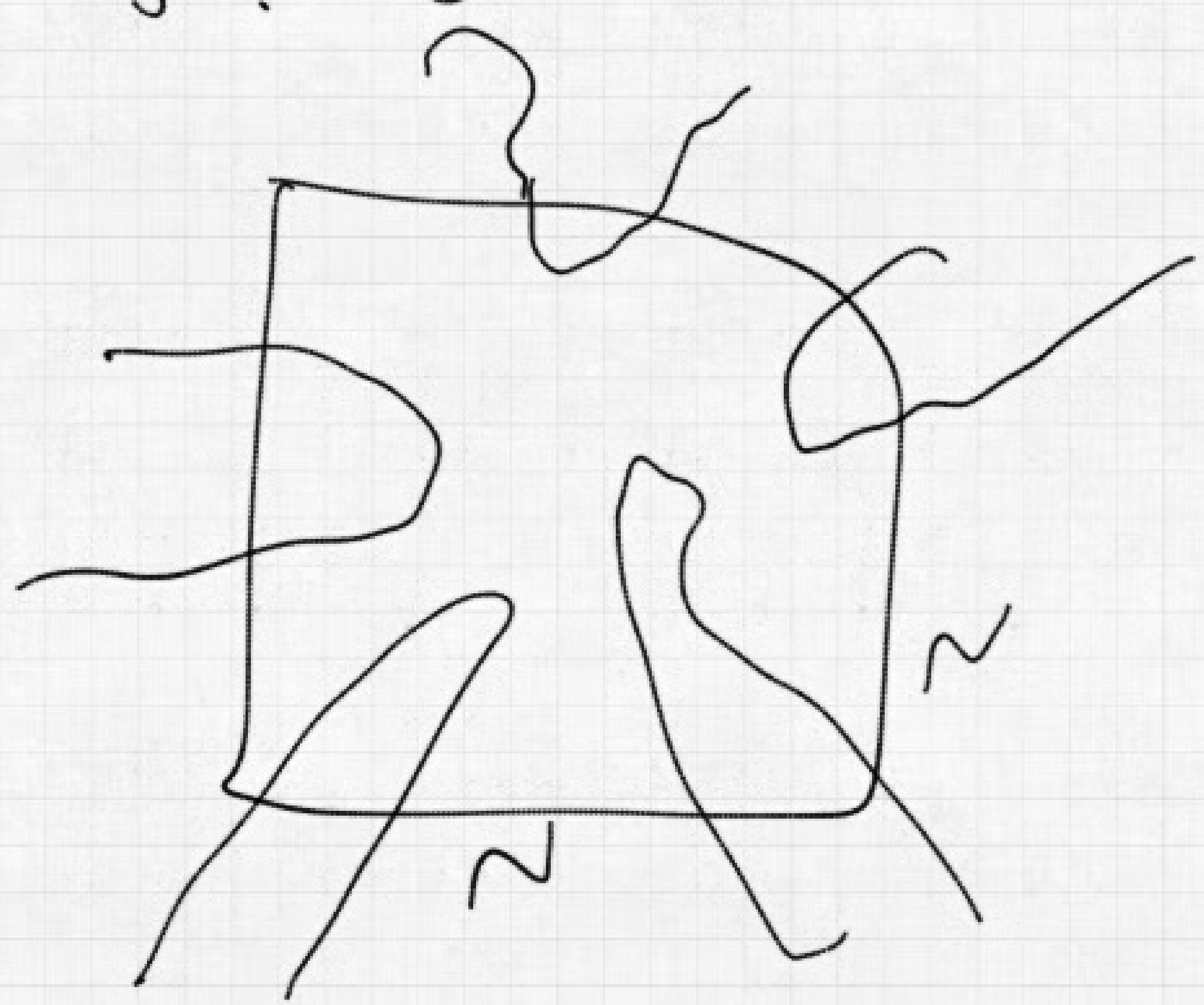
Proof: Let us assume $\mathbb{P}_p(E_5)=1$. Define $F_M=\{\exists$ 5 inf clusters

and each intersects

$[-M,M]^d\}$

Then $F_1 \subseteq F_2 \subseteq \ldots$ and $\bigcup_i F_i = E_5$. $\Rightarrow$ $\mathbb{P}_p(F_i) \xrightarrow{i \to \infty} 1$.

Choose N s.t. $\mathbb{P}_p(F_N) > 0$.



Let $\tilde{F}_N = \{$ 5 disjoint inf clusters outside $[-N,N]^d$ touching $\partial [-N,N]^d \}$.

$F_N \subseteq \tilde{F}_N \Rightarrow \mathbb{P}_p(\tilde{F}_N) > 0$. $\tilde{F}_N$ is ind of the configuration in $[-N,N]^d$, let G be the event all edges in $\square$ are open, $\mathbb{P}(\tilde{F}_N \cap G) = \mathbb{P}(\tilde{F}_N) \cdot \mathbb{P}(G) > 0$ but $\tilde{F}_N \cap G \subseteq E_1$

$\Rightarrow \mathbb{P}_p(E_1) > 0$!

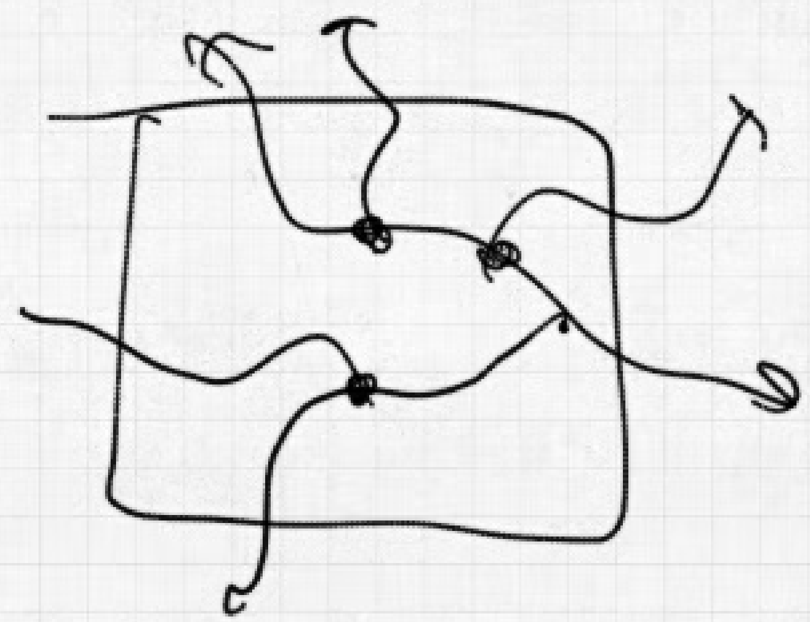
We are left with ruling out ∞ many infinite clusters.
(Burton Keane).

Let $Q = \#$ of inf clusters, assume $P_p(Q=\infty)=1$.

Call $z \in \mathbb{Z}^d$ a fork point if

1. $|C(z)| = \infty$.

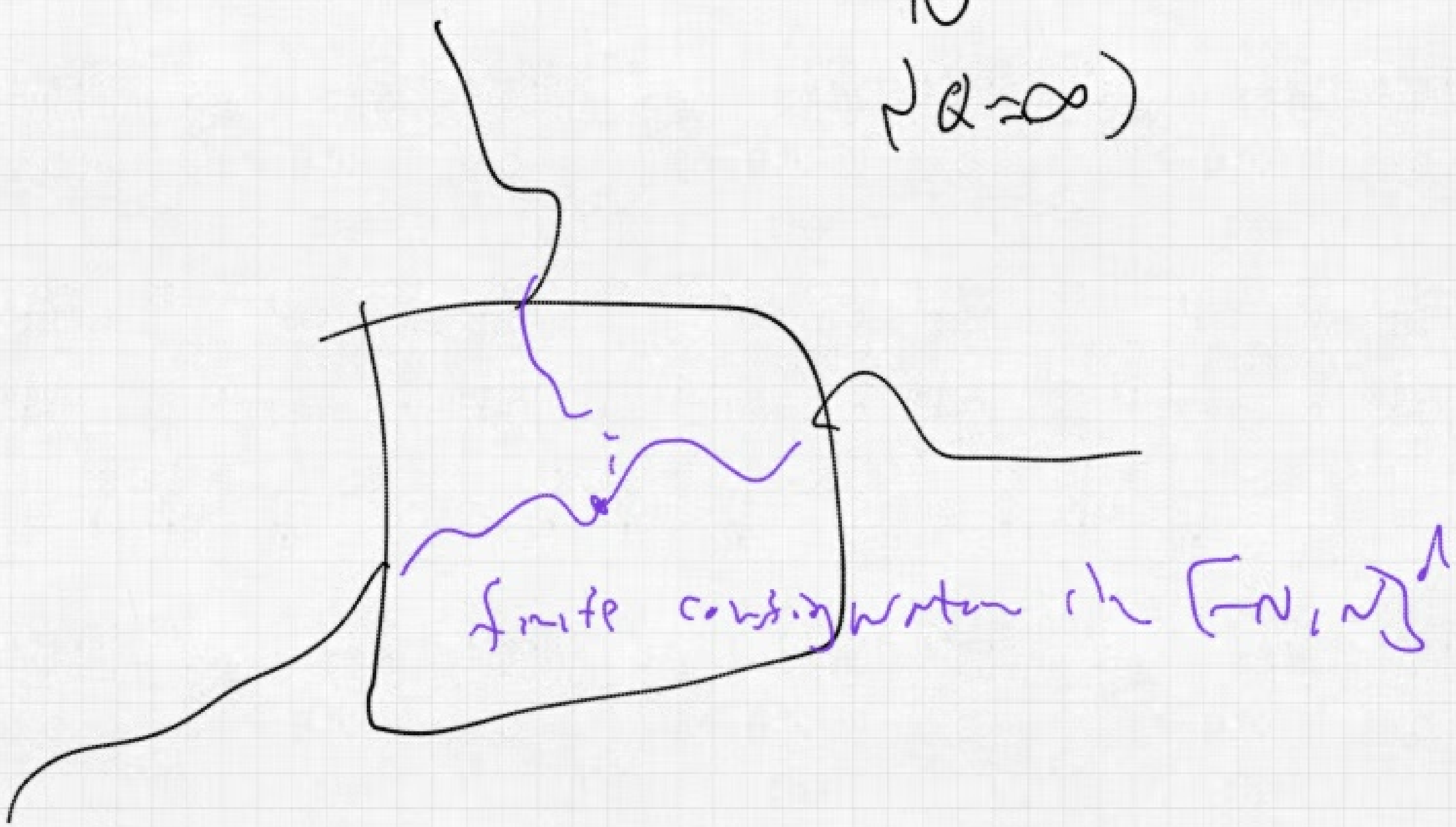
2. $C(z) \setminus \{z\}$ has no finite components and exactly 3 infinite components.



Lemma: If $\mathbb{P}_p(Q=\infty)=1$ then $\mathbb{P}_p(0 \text{ is a fork})>0$.

Proof: Let $F_n = \{$at least 3 inf clusters intersect $[-n,n]^d\}$.

Since $\bigcup_i F_i = \{Q \geq 3\} \supseteq \{Q=\infty\}$ we have $\mathbb{P}_p(F_i) \xrightarrow{i\to\infty} 1$. $\exists N$ $\mathbb{P}_p(F_N)>0$



$\blacksquare$

Lemma 1 For any configuration and $\forall N$, # fork points in $[-N,N]^d$ is at most $|\partial [-N,N]^d|$.

Finishing the proof given the Lemma: Let $\delta = P(0 \text{ is a fork point})$

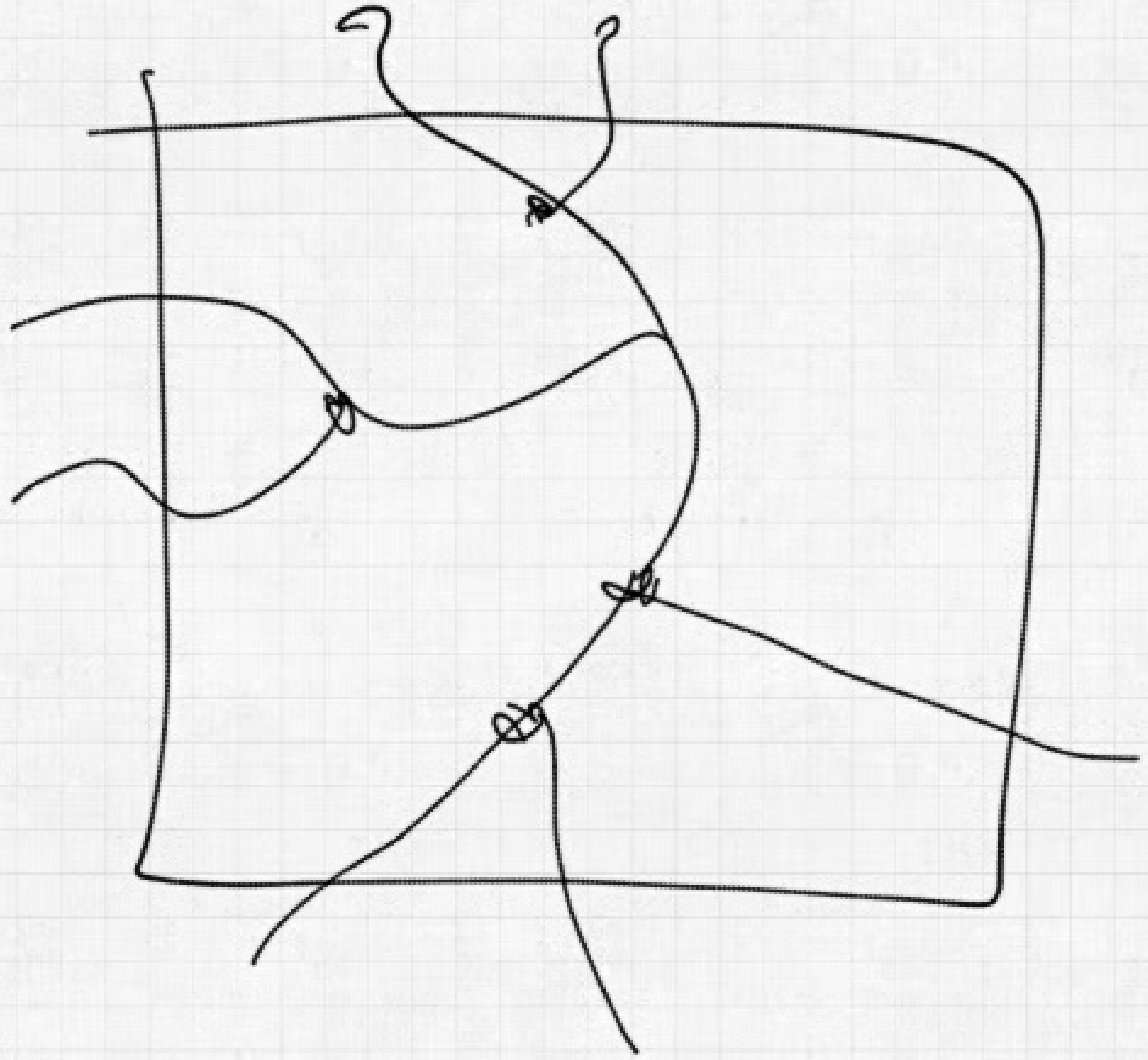
choose N large s.t. $\delta(2N+1)^d >> |\partial[-N,N]^d|$.

by Lemma # fork points $\leq |\partial[-N,N]^d|$

but $\mathbb{E}[\#\text{ fork points}] = \mathbb{E}\left[\sum_{x \in [-N,N]^d} \mathbb{1}_{x \text{ is a fork point}}\right] = \delta(2N+1)^d$ (

☒

proof of lemma:



1. $\forall$ finite set S of fork points in the same inf cluster $\exists$ one s.t removing it keeps all the rest in the same inf cluster.

2. (induction using 1) $\forall$ finite set S of fork points in the same inf cluster if we remove all elements of S we break the inf cluster to at least $|S|+2$ clusters.

Continuity of percolation function

Ex: $\theta_d(p)$ is a right continuous function of p on $[0,1]$ (Thm 2-5 Steif).

<u>Thm:</u> $\lim_{\pi \nearrow \tilde{p}} \theta(\pi) = \theta(\tilde{p})$.

<u>Proof:</u> Couple all percolation realizations on the same space.

$\{X(e) : e \in E^d\}$ iid $U(0,1)$. an edge is p-open if $X(e) < p$.

If $p_1 < p_2$ $\{e : e \text{ is } p_1 \text{ open}\} \subseteq \{e : e \text{ is } p_2 \text{ open}\}$.

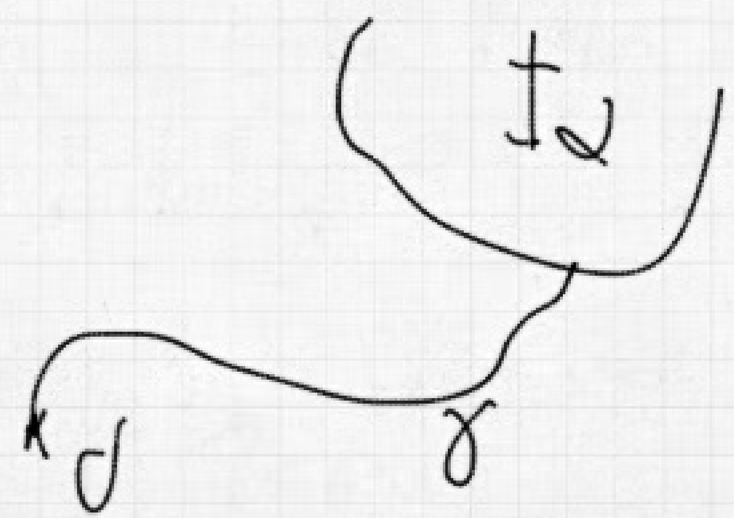
Let C_p - p-cluster of origin $\Rightarrow$ $C_{p_1} \subseteq C_{p_2}$ (increasing)

$$\lim_{\pi \nearrow \tilde{p}} \theta(\pi) = \lim_{\pi \nearrow \tilde{p}} P(|C_\pi| = \infty) = P\Big(\bigcup_{\pi < \tilde{p}} |C_\pi| = \infty\Big) = P\big(|C_\pi| = \infty \text{ for some } \pi < \tilde{p}\big)$$

Since $\{\exists \pi < \tilde{p} \mid |C_\pi| = \infty\} \subseteq \{|C_{\tilde{p}}| = \infty\}$ we need to show

$$P\left(\{|C_{\tilde{p}}| = \infty\} \cap \{|C_\pi| < \infty \ \forall \pi < \tilde{p}\}\right) = 0.$$

Let α: $p_c < \alpha < \tilde{p}$ then by uniqueness the inf α-cluster $I_\alpha \subseteq C_{\tilde{p}}$ a.s. If $0 \in I_\alpha$ choose $\pi = \alpha$. If not there is a path $\gamma : 0 \to I_\alpha$ $\tilde{p}$-open. Let



$$\mu = \max\{X(e) : e \in \gamma\}$$

$$\mu < \tilde{p}.$$

choose $\max\{\alpha, \mu\} < \pi < \tilde{p} \Rightarrow |C_\pi| = \infty$ ∎

FKG - inequality: For $\omega, \omega' \in \Omega = \{0,1\}^J$, there is a partial order $\omega \preceq \omega'$ if $\omega_i \leq \omega_i'$ $\forall i \in J$.

Definition: $f : \{0,1\}^J \to \mathbb{R}$ is increasing if $\omega \preceq \omega' \Rightarrow f(\omega) \leq f(\omega')$.

Example: $\exists$ open path between two vertices is increasing.

Theorem: Let f, g be increasing functions, then ($f, g \in L^2$)

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

Example: $x, y, z, w \in V(G)$, $A = \{\exists$ open path $x \leftrightarrow y\}$
$B = \{\ \cdots\ w \leftrightarrow z\}$

$$P(A \cap B) \geq P(A) P(B).$$

Proof: Assume J finite (approximate otherwise).

Induction on $|J|$. If $|J|=1$. Since f,g increasing

$$(f(\omega_1)-f(\omega_2))(g(\omega_1)-g(\omega_2)) \geq 0.$$

taking expectation

$$\mathbb{E}[f(\omega_1)g(\omega_1)] + \mathbb{E}[f(\omega_2)g(\omega_2)] - \mathbb{E}[f(\omega_2)g(\omega_1)] - \mathbb{E}[f(\omega_1)g(\omega_2)] \geq 0$$

$$\Rightarrow 2\mathbb{E}[fg] \geq 2\,\mathbb{E}[f]\,\mathbb{E}[g].$$

Assume true for $|J| = k-1$ and f,g functions of k variables.

$$\mathbb{E}\left[f(x_1,\ldots,x_n)\, g(x_1,\ldots,x_n)\right] = \mathbb{E}\left[\mathbb{E}\left[\,\text{---}\;\middle|\; x_1,\ldots,x_{n-1}\right]\right]$$

n=1 case

$$= \mathbb{E}\left[\mathbb{E}\left[f(x_1,\ldots,x_n)\,\middle|\,x_1,\ldots,x_{n-1}\right]\mathbb{E}\left[g(\;)\,\middle|\,x_1,\ldots,x_{n-1}\right]\right]$$

induction hypothesis

$$\overset{?}{=} \quad \mathbb{E}(f)\,\mathbb{E}(g)$$

∎

BK inequality: We would like to have a negative correlation inequality, $A \cap B$ is replaced with A and B occur on disjoint edge sets: $A \circ B$.

Let $\{(\Omega_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n$ be finite probability spaces and $(\Omega, \mathcal{F}, \mu)$ be the product.

Two configurations $\omega = (\omega_1, \ldots, \omega_n)$, $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_n) \in \Omega$ are said to be equal on $S \subset [n]$ if $\omega_i = \tilde{\omega}_i$ $\forall i \in S$.

Definition: Let $S \subset [n]$, and let $S^c = [n] \setminus S$. An event $A \in \mathcal{F}$ is said to occur on the set S in the configuration ω if A occurs using only the r.v over S.

$$A|_S = \{\omega : \forall \tilde{\omega} = \omega \text{ on } S \Rightarrow \tilde{\omega} \in A\}.$$

Two event $A_1, A_2 \in \mathcal{F}$ are said to occur disjointly, denoted $A_1 \circ A_2$ if there are two disjoint sets on which they occur.

$$A_1 \circ A_2 = \{\omega : \exists S_1, S_2 \subset [n],\ S_1 \cap S_2 = \emptyset,\ \omega \in A_1|_{S_1} \cap A_2|_{S_2}\}.$$

BK - Reimer: $\mu(A \circ B) \leq \mu(A)\mu(B)$

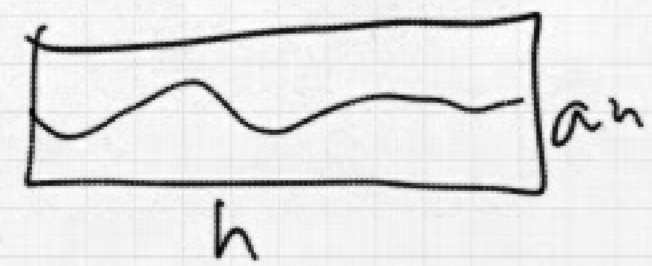
Example: 1. Let $G \subset \mathbb{Z}^d$ finite

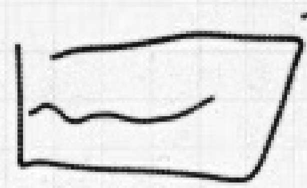
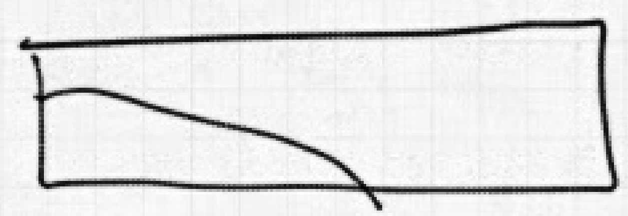
$$A_G(x,y) = \{\exists \text{ open path connecting } x \to y\}.$$

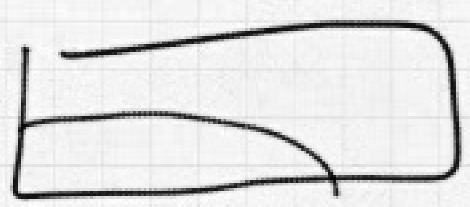
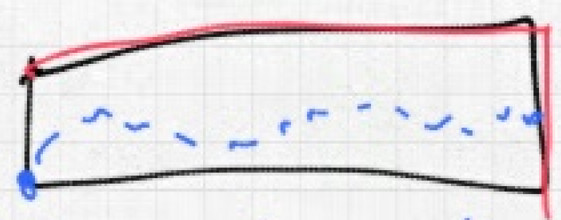
the event $A_G(x,y) \circ A_G(u,v) = \{\exists$ two edge disjoint open paths connecting $x \to y$ and $u \to v\}$.

$$P(A_G(x,y) \circ A_G(u,v)) \leq P(A_G(x,y))\, P(A_G(u,v)).$$

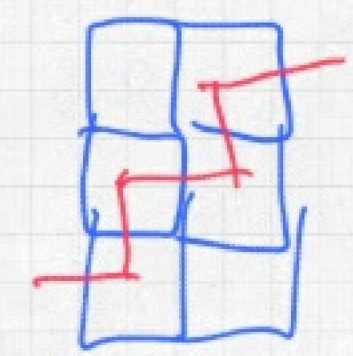
Distances in percolation:

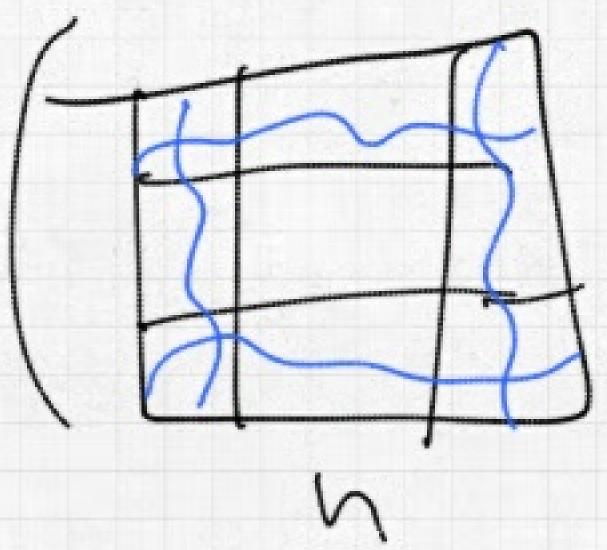
Lemma: $p > p_c(\mathbb{Z}^2)$, $P\Big($  $\Big) \xrightarrow[n\to\infty]{} 1$.

Proof: $P\Big($  $\Big) \geq P\Big($  $\Big)^2$ FKG

$P\Big($  $^c\Big) = P\Big($  $\Big) \leq e^{-cn}$.

dual closed path longer than n



By FKG $\quad \mathbb{P}\Big($  $\Big) \xrightarrow[n\to\infty]{} 1$

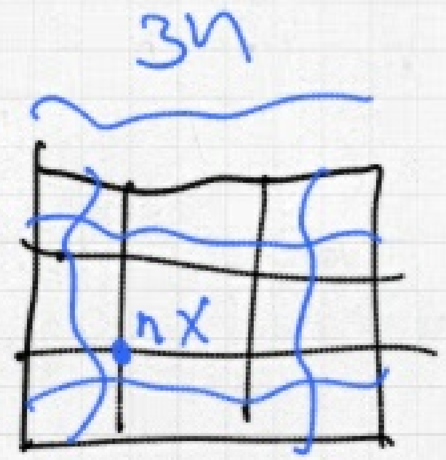
Domination by product measure: (Liggett, Schonmann, Stacey)

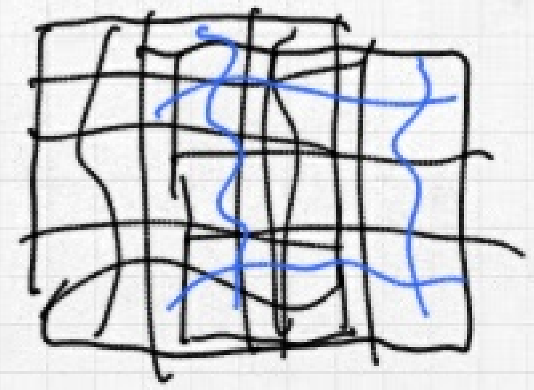
Let $(X_s)_{s\in\mathbb{Z}^d}$ be a collection of $\{0,1\}$ r.v. We say that (X_s) is k-dependent if $\{X_s : s\in A\}$, $\{X_s : s\in B\}$ are independent whenever $\mathrm{dist}(A,B) > k$.

Theorem [LSS 97] Let $d, k \geq 1$. $\exists$ non decreasing function $\pi : [0,1] \to [0,1]$ satisfying $\pi(\delta) \to 1$ as $\delta \to 1$ s.t. if (X_s) is k-dependent and $P(X_s = 1) \geq \delta$ $\forall x \in \mathbb{Z}^d$ then (X_s) is dominated (stochastic) from below by i.i.d percolation with parameter $\pi(\delta)$.

Definition: Let $Y = \{Y_s : s \in \Omega\}$, $Z = \{Z_s : s \in \Omega\}$ be families of $\{0,1\}$ variables. We say Y stochastically dominates Z, $Y \geq_{st} Z$, if

$$\mathbb{E}[f(Y)] \geq \mathbb{E}[f(Z)] \quad \forall \text{ bounded increasing } f : \{0,1\}^S \to \mathbb{R}.$$

Let $Y_x = \mathbb{1}\{$  $\}$

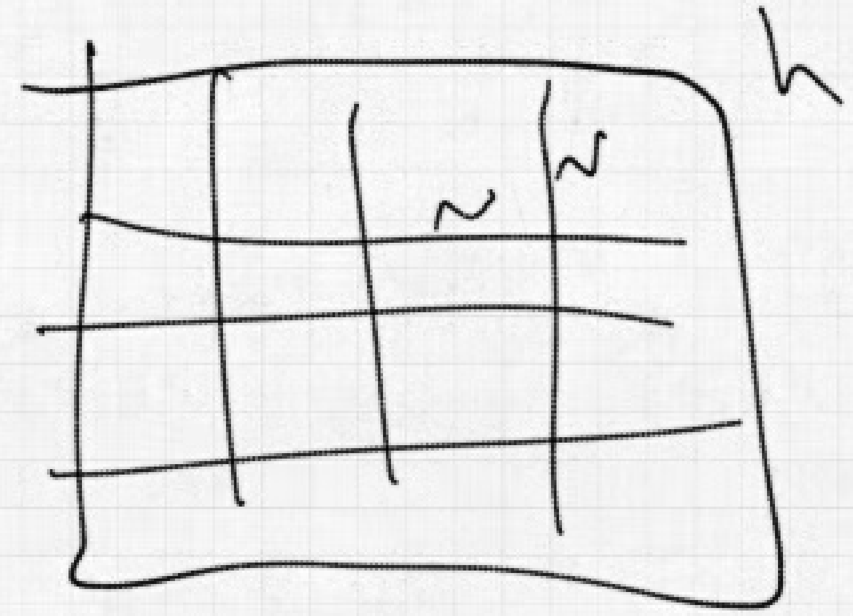


Thm: $\forall p > p_c(\mathbb{Z}^2)$ $\exists c_1, c > 0$ s.t. $\forall n$ large then

$$P_p(|C_d(n)| < c_1 n^d) < e^{-cn}$$

proof: Day 96 Lemma 1.2 $\exists p_c(\mathbb{Z}^2) < p^* < 1$ s.t. $\forall p > p^*$

$P_p(|C_{d(n)}(\omega)| < \tilde{c}_1 n^d) < e^{-cn}$.



whp

$C_d(n(\omega)) > \tilde{c}\left(\frac{n}{N}\right)^d$

by uniqueness ∎

$\forall\, x \sim y$

Theorem If $1-p < p_c$ then $d_w(x,y) < C$

w.h. positive prob

Proof

Let A be size of closed components

$$\mathbb{E}(A) \le \sum \|x-y\|_1 \cdot C$$

$C \rightarrow$ like size.

$$P\big(d(x,y) > \|x-y\|_1 + k\|x-y\|_1\big) \le \frac{\mathbb{E}\|A\|}{k\|x-y\|_1} = \frac{C}{k}$$

Left right crossing: $B(n) = [-n,n]^d$

Crossing of $B(n)$ is an open path of $B(n)$ joining some vertex x with $x_1 = -n$ to some y with $y_1 = n$. Denote $LR(n)$ the event of a crossing.

__Theorem__ If $\theta(p) > 1$ then $P_p(LR(n)) \to 1$ as $n \to \infty$.

__Proof__ (using uniqueness) Let $I(m) = \{C_\infty \cap B(m) \neq \emptyset\}$

Let $\varepsilon > 0$, choose m large s.t.

$$P_p(I(m)) > 1 - \varepsilon \quad \text{(why can we? ergodic theory)}.$$

For $n > m$ let $F_1, \ldots, F_{2d}$ be the faces of $B(n)$. If $I(m)$ occurs then some vertex in $B(m)$ is connected to a face.

$$1 - \mathbb{P}_p(\mathbb{I}(m)) \geq 1 - \mathbb{P}_p\left(\bigcup_{i=1}^{2d} \{B(m) \leftrightarrow F_i \text{ in } B(n)\}\right)$$

$$= \mathbb{P}_p\left(\bigcap_{i=1}^{2d} \{B(m) \leftrightarrow F_i \text{ in } B(n)\}^c\right)$$

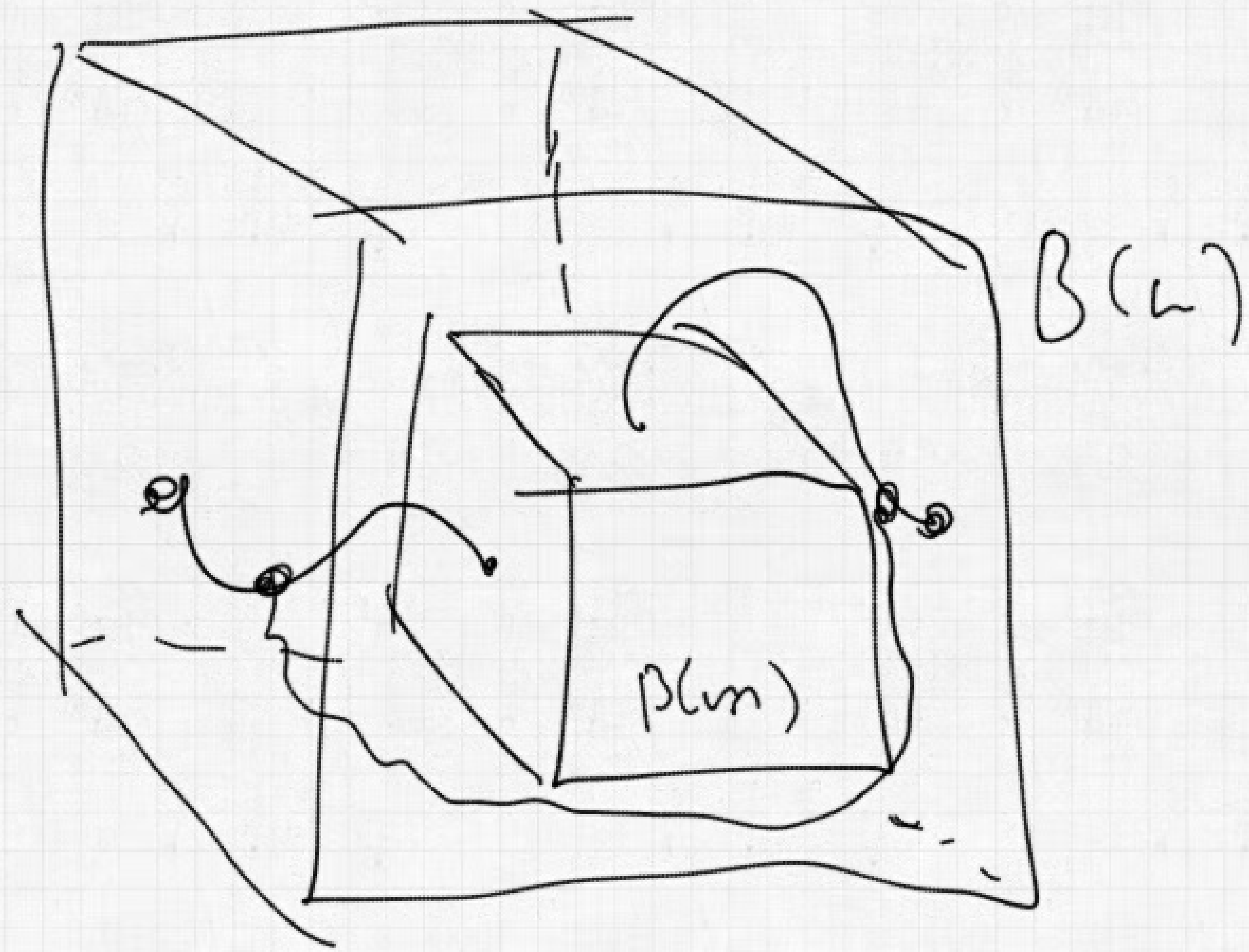
$$\overset{\text{FKG}}{\geq} \left(1 - \mathbb{P}_p(B(m) \leftrightarrow F \text{ in } B(n))\right)^{2d}$$

some chosen face.

$$\Rightarrow \mathbb{P}_p(B(m) \leftrightarrow F \text{ in } B(n)) \geq 1 - \varepsilon^{1/2d}$$

$$\Rightarrow \mathbb{P}_p(\{B(m) \leftrightarrow F_L \text{ in } B(n)\} \cap \{B(m) \leftrightarrow F_R \text{ in } B(n)\}) \geq (1 - \varepsilon^{1/2d})^2$$

left face



Let $A_{m,n} = \{\exists$ two vertices of $\partial B(m)$ in disjoint open clusters of $B(n)$ both intersect $\partial B(n)\}$.

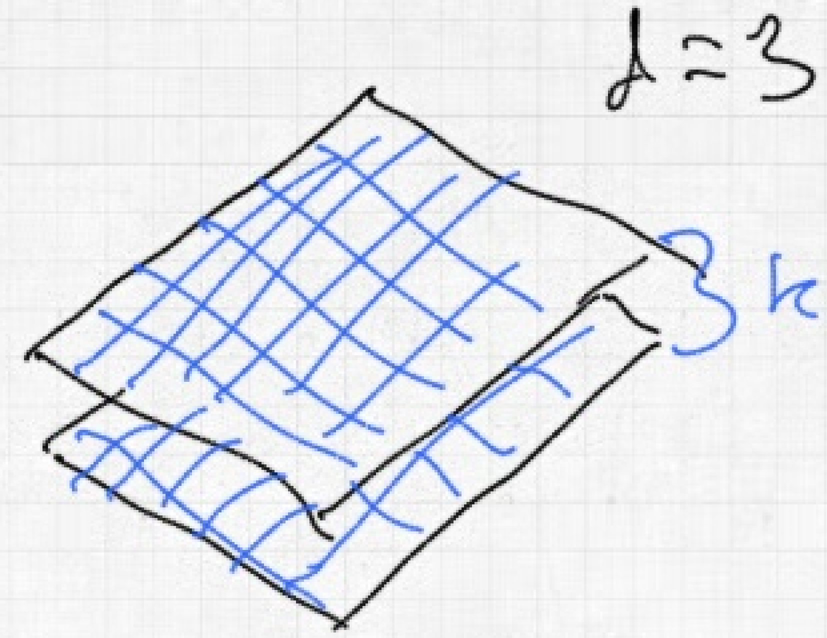
Since $A_{m,n} \supseteq A_{m,n+1}$ $\quad P_p(A_{m,n}) \xrightarrow[n\to\infty]{} P_p(\exists$ two vertices of $\partial B(m)$ in disjoint inf clusters$) = 0$

$\Rightarrow P_p(LR(n)) \geq (1-\varepsilon^{\|\ldots\|^2}) - P_p(A_{m,n})$ ■

Percolation in Slabs:

For $d \geq 3$ define the 2d-slab of thickness k

$$S_k = \mathbb{Z}^2 \times \{0,1,\dots,k\}^{d-2}$$



Since $S_k \subseteq S_{k+1} \subseteq \dots \mathbb{Z}^d$

$p_c(S_k) \geq p_c(S_{k+1}) \geq p_c$. Bounded decreasing thus converges

$$p_c^{slab} = \lim_{k \to \infty} p_c(S_k)$$

<u>Theorem</u>: Grimmett, Marstrand 1990:

$\forall d \geq 3 \quad p_c^{slab} = p_c$.

Let $S_n(L) = [-n,n]^2 \times [0,L]^{d-2}$

$T_n(L) = [-n,n]^{d-1} \times [0,L]$.

Lemma: Let $d \geq 3$ and $p > p_c$. There exists a positive integer L and a $\delta(p,L) > 0$ s.t

① $\mathbb{P}_p(x \leftrightarrow y \text{ in } S_n(L)) \geq \delta \quad \forall x,y \in S_n(L)$ and $n \geq 1$.

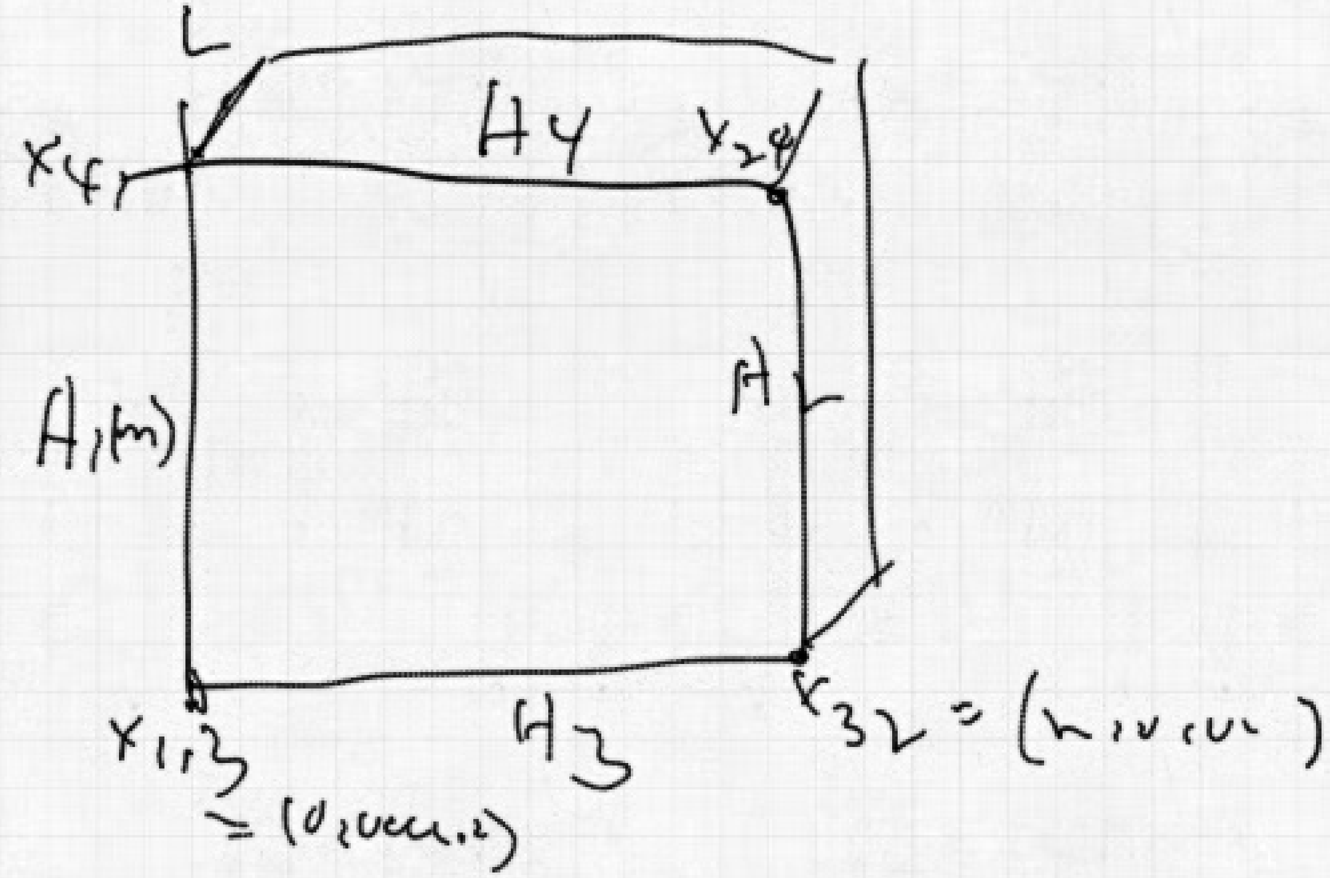
② $\mathbb{P}_p(x \leftrightarrow y \text{ in } T_n(L)) \geq \delta \quad \forall x,y \in T_n(L)$ and $n \geq 1$ (HV Grimmett 7.78 contrival args).

<u>Proof:</u> Since $p > p_c$, by Grimmett Marstrand $\exists L$ s.t.

$$p > p_c(\mathbb{Z}_+^2 \times [0,L]^{d-2}).$$

Let $p > p' > p_c(\mathbb{Z}^2 \times [0,L]^{d-2})$. Denote $\theta = \mathbb{P}_{p'}\left(0 \leftrightarrow \infty \text{ in } \mathbb{Z}_+^2 \times [0,L]^{d-2}\right) > 0$

Denote on $U_n(L) = [0,n]^2 \times [0,L]^{d-2}$



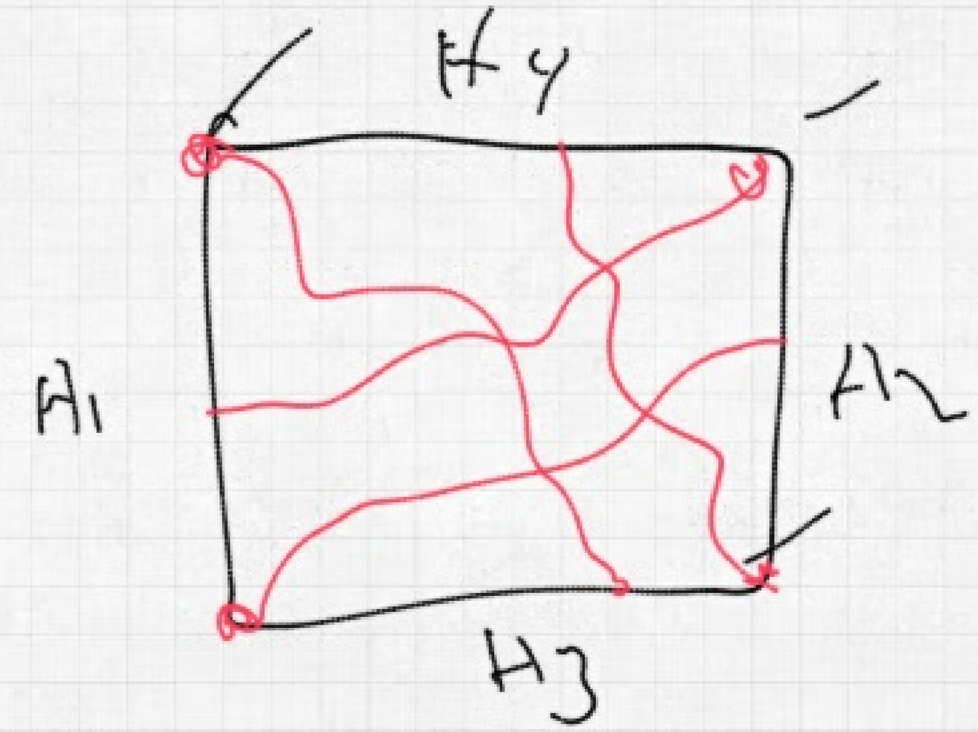
$$\theta \le \mathbb{P}_{p'}\left(0 \leftrightarrow H_2(n) \cup H_4(n) \text{ in } U_n(L)\right)$$

$$\le \mathbb{P}_{p'}\Big[0 \leftrightarrow H_2(n) \text{ in } U_n(L)\Big] + \Big[0 \leftrightarrow H_4(n)\Big]$$

$$\Rightarrow \mathbb{P}_{p'}\left(0 \leftrightarrow H_2(n) \text{ in } U_n(L)\right) \ge \tfrac{1}{2}\theta$$

By rotation invariance the probability of all the following events $\geq \frac{1}{2}\theta$

$A_{13} = \{ X_{13} \leftrightarrow H_2(h) \text{ in } U_m(L) \}$



$A_{32} = \{ X_{32} \leftrightarrow H_4(h) \text{ in } U_m(L) \}$

FKG

$P_p(A_{13} \cap A_{32} \cap A_{24} \cap A_{41}) \geq (\frac{1}{2}\theta)^4$

there is a set of at most $4(d-2)L$ that if open $X_{13} \leftrightarrow$

$X_{32} \leftrightarrow X_{34} \leftrightarrow X_{41}$ in $U_m(L) \Rightarrow \exists \delta_1(\theta, L) > 0$ s.t.

$P_p(X_{13} \leftrightarrow \quad \leftrightarrow X_{41} \text{ in } U_m(L)) \geq \delta_1 \quad \forall m \geq 1.$

Now by FKG it is enough to show

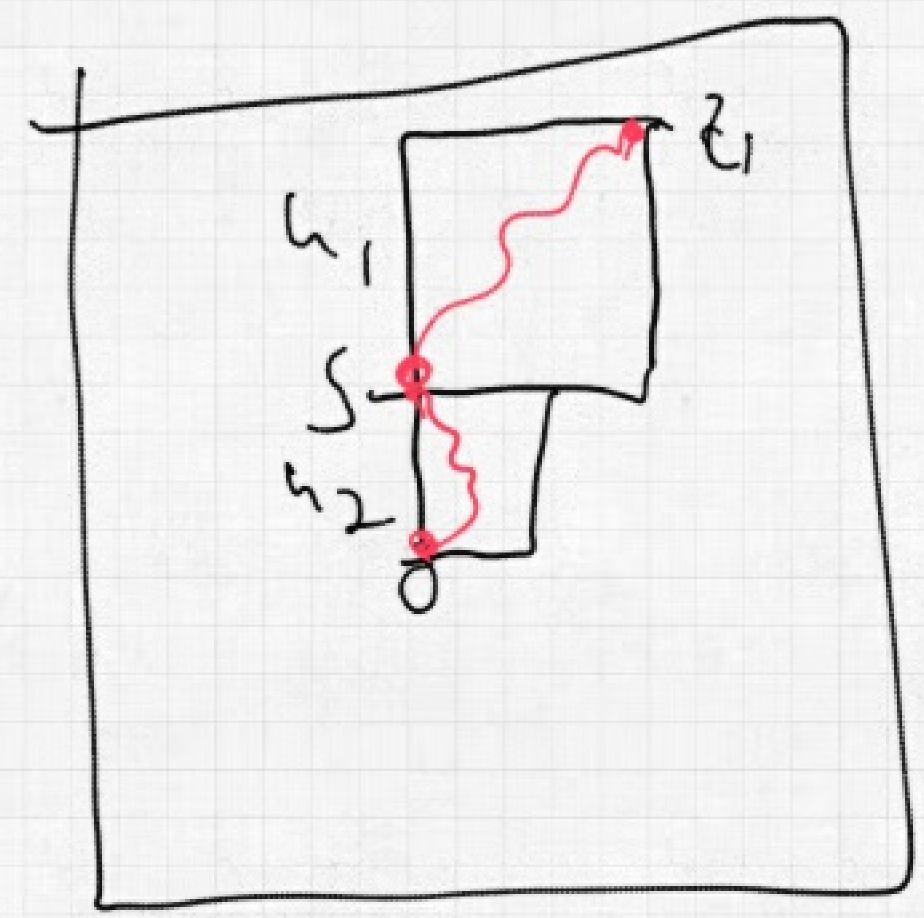
$P_p(0 \leftrightarrow z \text{ in } S_L(L)) \geq \delta_2 > 0 \quad \forall z \in S_n(L)$.

Let $z = (z_1, z_2, \ldots, z_d) \in S_n(L)$. assume $^{\text{wlog}}$ $0 \leq z_1 \leq z_2$

Define the boxes,

$$U_1 = [0, z_1] \times [z_2 - z_1, z_2] \times [0, L]^{d-2}$$

$$U_2 = [0, z_2 - z_1]^2 \times [0, L]^{d-2}$$



by FKG

$(0, z_2 - z_1, 0, \ldots, 0)$

$$P_p(0 \leftrightarrow z \text{ in } S_n(L)) \geq P_p(z \leftrightarrow y \text{ in } U_1)\, P_p(y \leftrightarrow 0 \text{ in } U_2)$$

also

$$P_p(z \leftrightarrow y \text{ in } h_1) \geq p^{(d-2)L} P_p(z' \leftrightarrow y \text{ in } h_1) \text{ where}$$

$z' = (z_1, z_2, \ldots)$ (path of length $(d-2)L$ connecting them,

$$\Rightarrow P_p(z \leftrightarrow 0 \text{ in } S_n(L)) \geq p^{(d-2)L} \delta_1^2$$

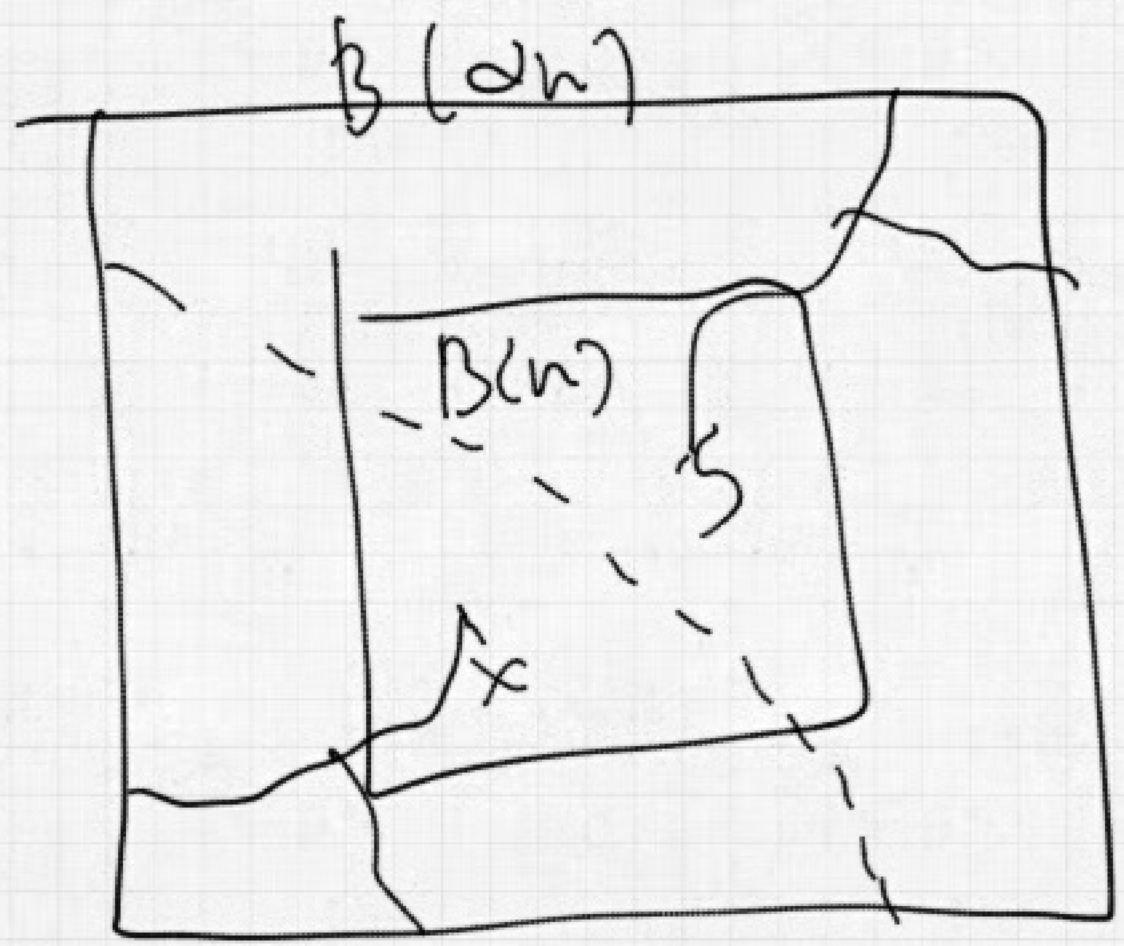
∎

Lemma 7.7

Lemma (H W) let $d \geq 3$, $p > p_c$ $\exists\, \zeta = \zeta(p) > 0$ s.t

$$P_p\left(x \leftrightarrow \partial B(\alpha n),\ y \leftrightarrow \partial B(\alpha n),\ x \not\leftrightarrow y \text{ in } B(\alpha n) \right) \leq e^{-n(\alpha-1)\zeta}$$

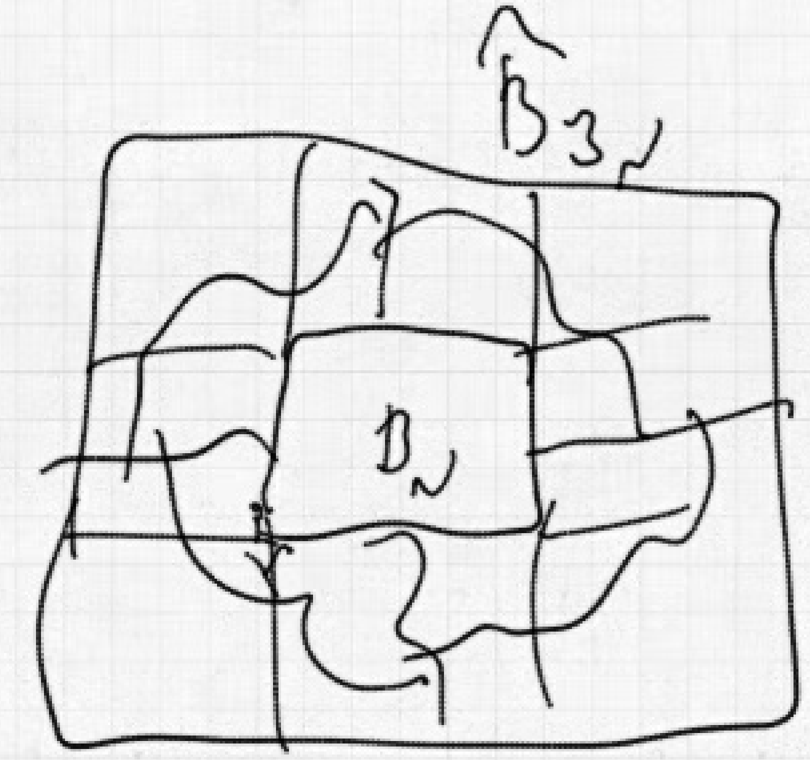
$\forall\, x, y \in B(n)$, $n \geq 1$, $\alpha > 1$.



Chemical Distance:

Denote $B_N(x) = x + \mathbb{Z}^d \cap [0, N]^d$

and $\tilde{B}_{3N}(x) = x + \mathbb{Z}^d \cap [-N, 2N]^d$



Let $G_N(x)$ be the event:

(1) $\forall x$ neighbor of y the side of the block $B_N(Ny)$ adjacent to $B_N(Nx)$ is connected to the opposite side of $B_N(Ny)$ by an open path.

(2) every two open paths connecting $B_N(Nx)$ to $\partial \tilde{B}_{3N}(Nx)$ are connected in $\tilde{B}_{3N}(Nx)$.

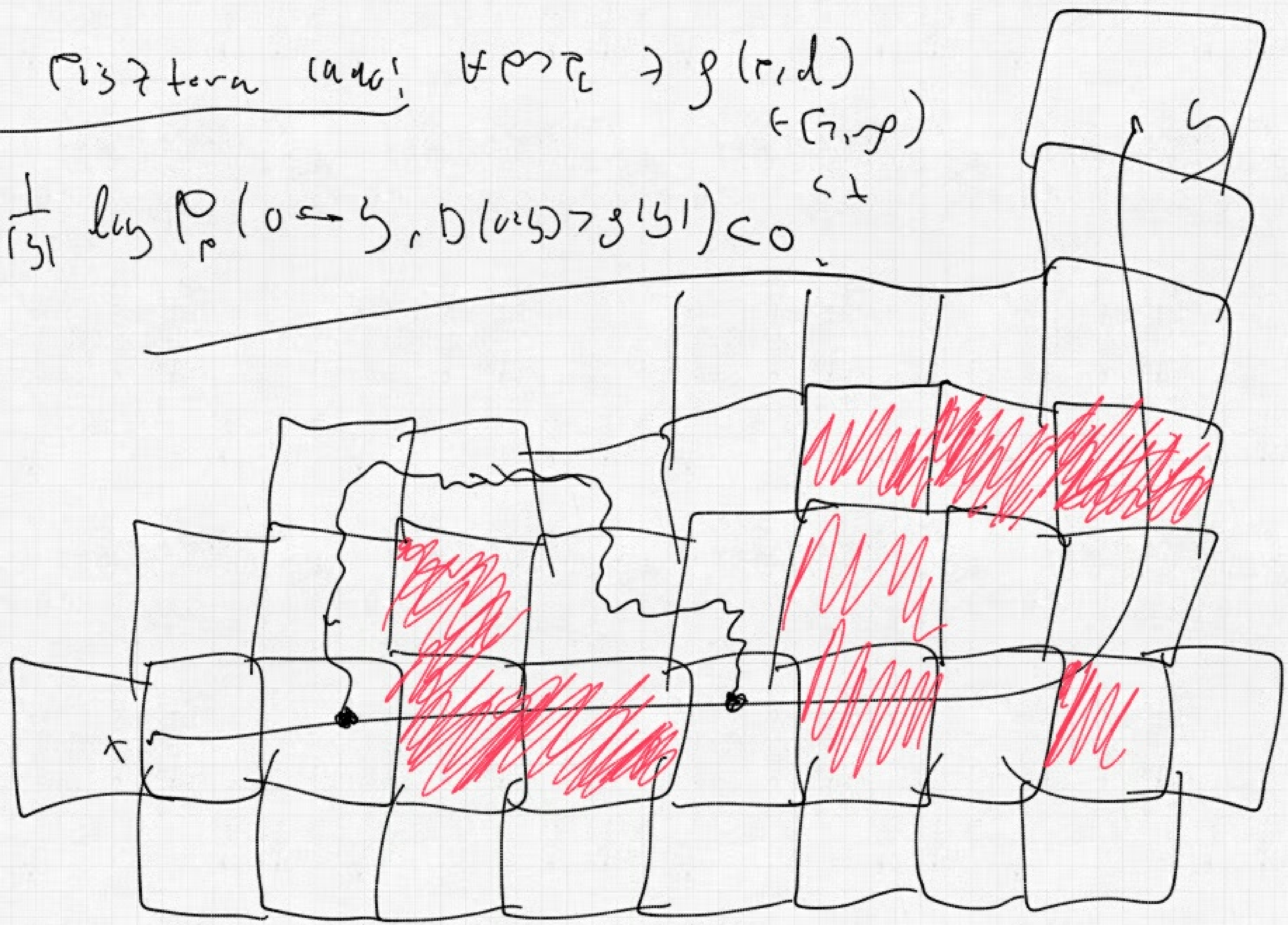
We saw $\mathbb{P}_p(G_N(0)) \xrightarrow[N\to\infty]{} 1$ thus $\{1_{G_N(x)} : x \in \mathbb{Z}^d\}$
dominated from below by i.i.d percolation with
parameter $\eta_N(p)$,

for the points x, y we want to find

an open path connecting them. If $\|x-y\|_1 = n$
one has at most n/N boxes of size N intersecting
an L_1 path connecting them.

Antal Pisztora cond.: $\forall p > p_c \ \exists \rho(p,d) \in [1,\infty)$

$$\lim_{|y|\to\infty} \frac{1}{|y|} \log \mathbb{P}_p\left(0 \leftrightarrow y,\ D(0,y) > \rho |y|\right) < 0$$



Let $\{C_i\}$ be the collection of bad connected macro-clusters intersecting γ.

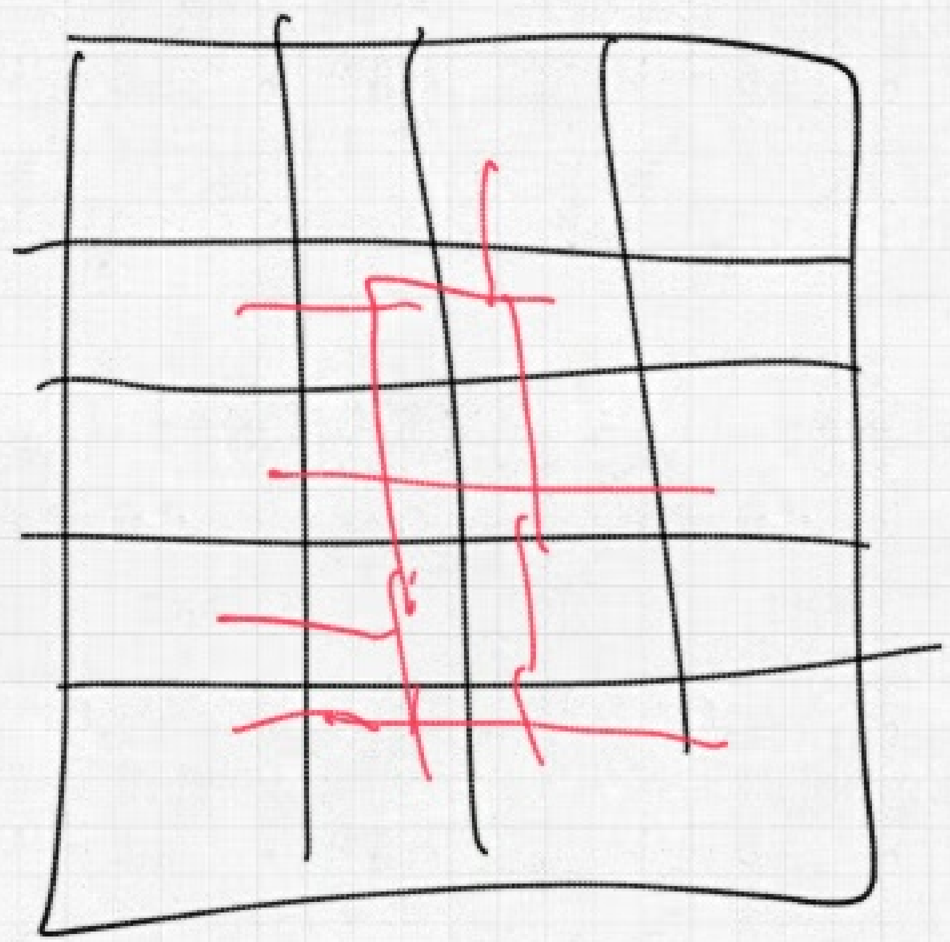
$$|\gamma'| \leq (3N)^d \left(|\gamma| + \sum_i |C_i| \right)$$

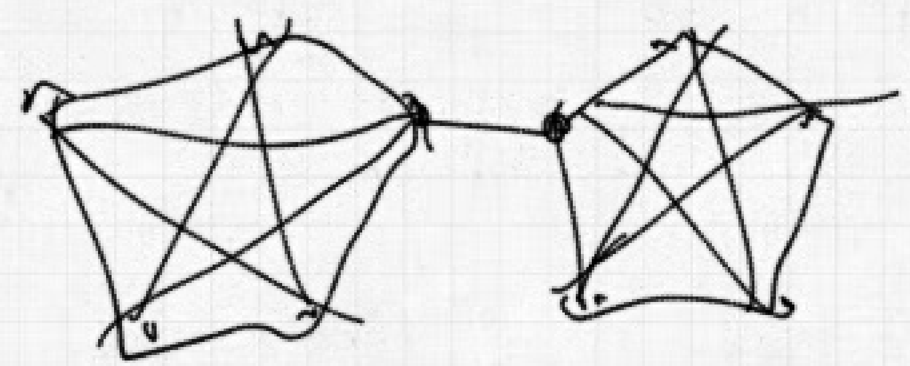
$$\sum_i |C_i| \underset{BK}{\leq} \sum_i |\tilde{C}_i|$$ then Large deviations.

($\tilde{C}_i$: i.i.d. copies of [illegible] of 0.)

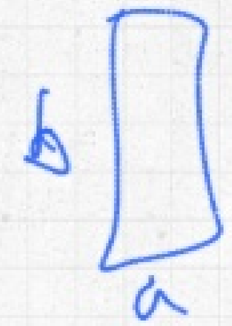
Isoperimetry:

for a finite graph G $\quad \varphi_G = \inf \left\{ \frac{|\partial A|}{|A|} : |A| < \frac{|G|}{2} \right\}$





Isoperimetry of $\mathbb{Z}^2$.



$2a+2b \geq 4\sqrt{ab}$

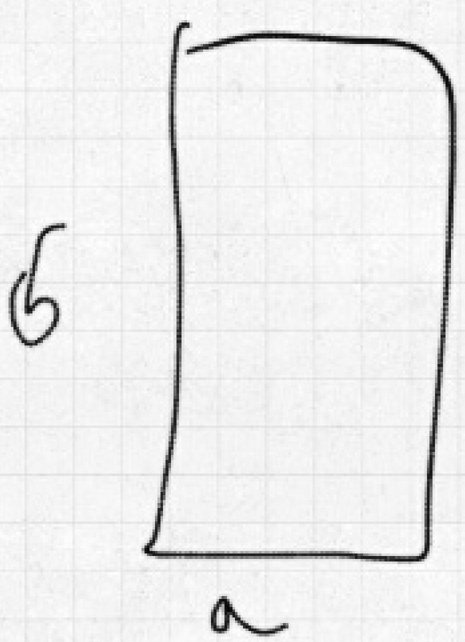
$|\partial A| \geq |\partial \square(A)|$

$\geq |\square(A)|^{n/d} \overset{?}{\geq} |A|^{n/d}$

larger value smaller boundary. Contract to have the same value the boundary will be smaller still.

$(x^n)' = nx^{n-1}$

we get a rectangle with smaller cheeger.



$ab = n^2$ $\qquad a = \frac{n^2}{b}$

$|\partial A| = 2a+2b \overset{?}{\geq} 4n$

$b > a$

$(b > n)$

$2a+2b = 2\frac{n^2}{b} + 2b$

$(\;)' = -2n^2 b^{-2} + 2 = 0$

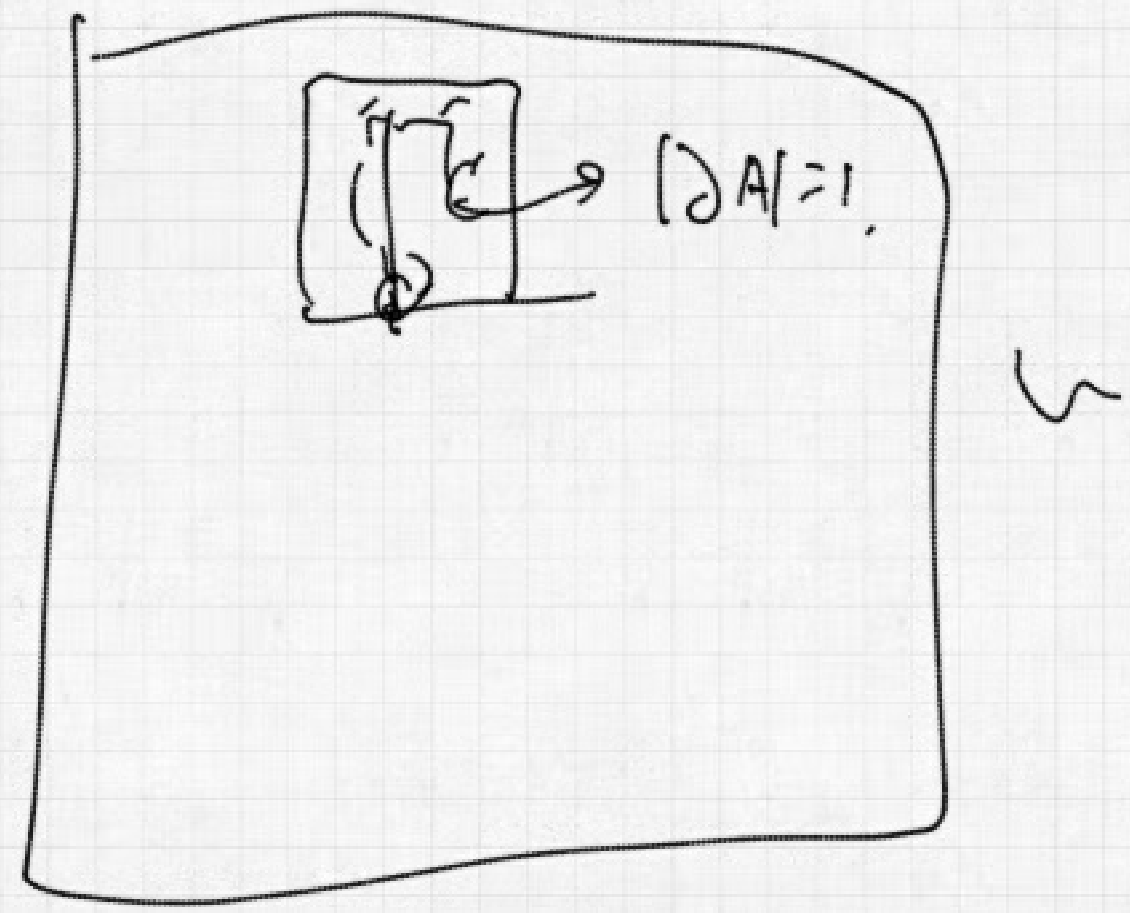
$b = n$ ∎

What about Percolation?

If we look at a box

we need to restrict ourselfs

to "big" sets.



BBHK 08

Theorem: $\forall d \geq 2$, $p > p_c(d)$ $\exists c_1(p,d)$, $c_2(d,p)$ and an a-s

finite r.v R_0 s.t $\forall R > R_0$ and each ω-connected A

$$A \subset C_\infty \cap [-R,R]^d \text{ and } |A| \geq (c_1 \log R)^{d/(d-1)}$$

we have

$$|\partial^\omega A| \geq c_2 |A|^{d-1/d}.$$

Given a finite set

$\Lambda \subset \mathbb{Z}^d$, let $\Lambda^{(N)} = \{x \in \mathbb{Z}^d : \Lambda \cap B_N(Nx) \neq \emptyset\}$

$\overline{\Lambda}^{(N)}$ complement of the inf comp of $(\Lambda^{(N)})^c$,

inner site boundary $\partial^* \Delta = \{x \in \Delta : \exists y \in \mathbb{Z}^d \setminus \Delta$ with $|x-y| = 1\}$,

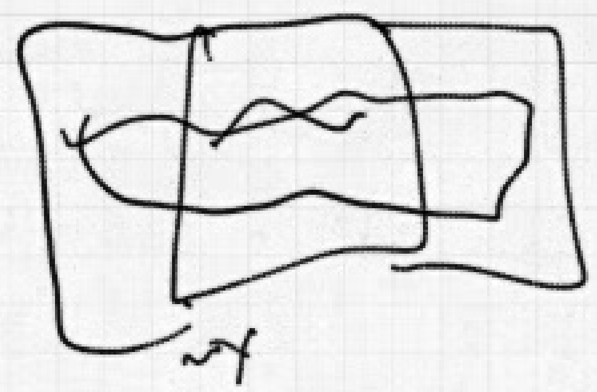
Lemma: For $N \in \mathbb{N}$, let $\Lambda \subset \mathbb{Z}^d$ be ω-connected with $\overline{\Delta}^N = \Delta$ and $\operatorname{diam}_{l_\infty} \Lambda \geq 3N$. If $|\partial^\omega \Lambda| < \frac{1}{2 \cdot 3^d} |\partial^* \Delta|$ then

$$\left|\{x \in \partial^* \Delta : G_N x f \text{ occurs}\}\right| > \frac{1}{2} |\partial^* \Delta|.$$

Proof:

If $x \in \partial^* \Delta$ then $\Lambda \cap B_N(Nx) \neq \emptyset$. We claim $\forall x \in \partial^* \Delta$

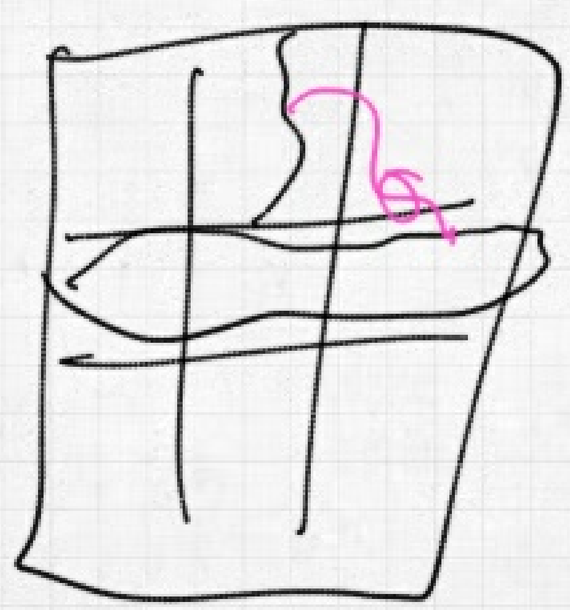
$G_N(x) \subset \{ \tilde{B}_{3N}(Nx)$ contains an edge in $\partial^\omega \Delta \}$



If $G_N(x)$ occurs, by $\operatorname{diam} \Lambda > 3N$ $B_N(Nx)$

is connected to $\partial \tilde{B}_{3N}(Nx)$ by an open path in Λ.

by definition of $G_N(x)$ $\exists$ another crossing to a $B_N(Ny) \cap \Lambda = \emptyset$.



these paths are connected by a path in $\tilde{B}_{3N}(Nx)$ which must contain a boundary edge, Since each edge in $\partial^w \Lambda$ belongs to at most 3^d $\hat{B}_{3N}(\cdot)$ cubes, with $x \in \partial^* \Delta$ the # of of $x \in \partial^* \Delta$ where $G_N(x)$ occurs is bounded by $3^d |\partial^w \Delta|$,

$$\Rightarrow \quad |\partial^* \Delta| - |\{x \in \partial^* \Delta,\ G_N(x)^c \text{ occurs}\}| \le 3^d |\partial^w \Delta| \underset{\text{assumption}}{\le} \tfrac{1}{2} |\partial^* \Delta|$$

∎

<u>Proof of main Theorem</u>: Let $c_4 = (2 \cdot 3^d)^{-1}$. Fix $\Delta \subset \mathbb{Z}^d$ finite connected with connected complement. Suppose Δ is w-connected with $\bar{\Delta}^N = \Delta$. ($|\Delta| > t^{d/(d-1)} \ge (3N)^d$) (for the lemma).

Then $|\Delta| \geq N^{-d} |\mathbf{\Delta}|$. standard isoperimetry on $\mathbb{Z}^d$

$|\partial^* \Delta| \geq c_5 |\Delta|^{d-1/d} \geq c_5 N^{1-d} |\mathbf{\Delta}|^{d-1/d}$. Set $c_2 = c_4 c_5 N^{1-d}$

$\{ |\partial^w \mathbf{\Delta}| < c_2 |\mathbf{\Delta}|^{d-1/d} \} \subset \{ |\partial^w \Delta| < c_4 |\partial^* \Delta| \}$.

And $|\mathbf{\Delta}| \geq t^{d/d-1} \Rightarrow |\partial^* \Delta| \geq c_5 N^{1-d} t$.

Since $\{\mathbb{1}_{(N,\cdot)}\}$ dominates dual percolation with $\eta_N(p) = 1 - \varepsilon_N$

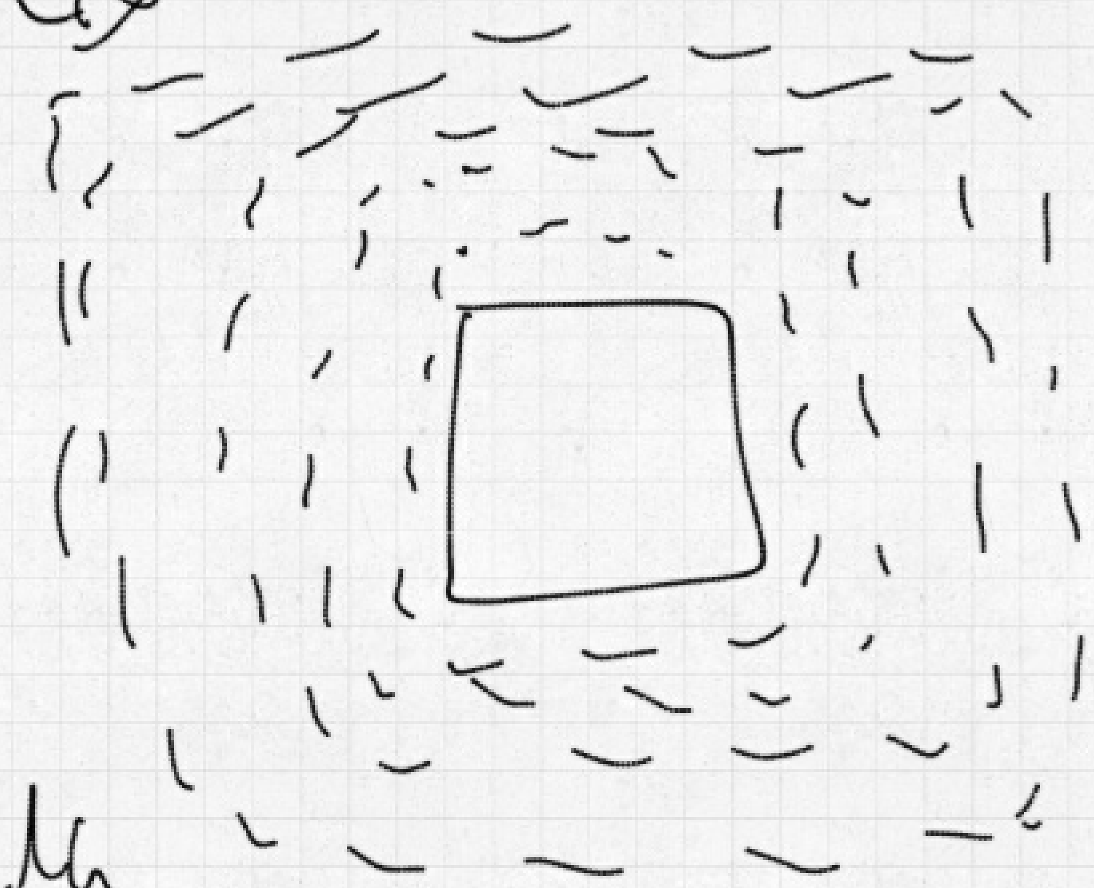
$$\mathbb{P}\left(\exists \mathbf{\Delta} \ni 0, \ w\text{-connected}, |\mathbf{\Delta}| \geq t^{1/d-1}, \tilde{\Delta}^N = \Delta, \ |\partial^w \mathbf{\Delta}| < c_2 |\mathbf{\Delta}|^{d-1/d} \right)$$

$$\leq \mathbb{P}\left(\sum_{x \in \partial^* \Delta} \mathbb{1}_{(N)}(x) \leq \tfrac{1}{2} |\partial^* \Delta| \right) \leq 2^{|\partial^* \Delta|} \varepsilon_N^{(\frac{1}{2})|\partial^* \Delta|}$$ union bound.

Now choose ε_r small enough s.t. ε_r^n is smaller than H of sets with boundary h. ∎

RW on percolation cluster on $\mathbb{Z}^d$

$$P_\omega[X_{n+1}=z \mid X_n=y] = \begin{cases} \frac{1}{2d} & \|z-y\|_1 = 1,\ z \in C_\infty \\ 1 - \frac{\deg_\omega(y)}{2d} & z = y \\ 0 & \text{otherwise} \end{cases}$$



Let $\phi(r) = \inf\left\{ \frac{|\partial S|}{|S|} : |S| \le r \right\}$

Theorem (Morris, Peres 2005) If $n \geq 1 + \int_4^{4/\varepsilon} \frac{4\,du}{u\,\phi^2(u)}$

then $p^n(x,y) \leq \varepsilon$.

$\varepsilon = \frac{1}{n^{d-2}}$

If $\forall_{r>} |S| > n^{1/3}$ $\quad |\partial S| > c|S|^{d-1/d}$

$$\frac{|\partial S|}{|S|} = \frac{|\partial S|}{|S|^{d-1/d}\,|S|^{1/d}} \geq \frac{c}{|S|^{1/d}} \geq \frac{c}{r^{1/d}}$$

$$\int_{n^{1/3}}^{4/\varepsilon} \frac{du}{u\,\phi^2 u} = \int_{n^{1/3}}^{4/\varepsilon} \frac{du}{u\,\frac{c}{u^{2/d}}} = \frac{1}{c}\int_{n^{1/3}}^{4/\varepsilon} u^{2/d-1}\,du \approx \frac{1}{\varepsilon^{2/d}}$$

$$\Rightarrow \text{ if } \varepsilon = \frac{C}{N^{d/2}} \Rightarrow p^n(x,y) \leq \frac{C}{N^{d/2}}$$

If $|S| < n^{1/3}$ $\quad \frac{|\partial S|}{|S|} > \frac{1}{n^{1/3}} \Rightarrow \int_1^{n^{1/3}} \frac{du}{u\,\phi^2(u)} \leq n^{1/3} \cdot n^{2/3} \simeq n$

transience for $d \geq 3$

$$P(X_n = (0, \ldots) \text{ i.o.}) = 0$$

since $\sum p^n(0,0) \approx \sum \frac{c}{n^{d/2}} < \infty$ $d \geq 3$

Invariance principle: Consider

$$B_n(t) = \frac{1}{\sqrt{n}} \left[X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor}) \right]$$

$\forall d \geq 2,\ p > p_c(d)$

Theorem: The law of $(B_n(t) : 0 \leq t \leq T)$ on $(C[0,T], W_T)$ converges weakly to an isotropic Brownian motion with positive diffusion constant (p, d)