Subcollection Sum Divisibility Theorems

Problems

The goal of this problem set is to determine the values of k and n for which the following statement is true or false:

$P(k,n)$: For every collection, S, of n integers there is a subcollection, T, of k integers whose sum is divisible by k.

The first group of problems give examples with values of k and n where $P(k,n)$ is true. The second group of problems will determine all values of k and n for which $P(k,n)$ is false by giving counterexamples. And the third group of problems will determine all values of k and n for which $P(k,n)$ is true. First some definitions.

Definitions: A *collection* is a list of elements in which elements are allowed to repeat and the order does not matter. So for example, $S = \{2,3,2,5,4,7,3\}$ is a collection of 7 numbers. Since the order of the elements does not matter, we can also write $S = \{2,2,3,3,4,5,7\}$ where we have written the numbers in ascending order. The number of times an element repeats is its *multiplicity*. A collection T is a *subcollection* of S (written $T \subseteq S$) if every element of T is an element of S and the multiplicity of each element in T is less than or equal to its multiplicity in S. Thus for the same example, $\{2,2,3,5\} \subseteq S$ but $\{2,3,3,3,4,5\} \not\subseteq S$. We write $|S|$ for the number of elements in S and ΣS for the sum of the elements in S. So for the example S above, we have $|S| = 7$ and $\Sigma S = 26$. We also define $S + r$ to be the collection where r has been added to each element of S, and rS to be the collection where each element of S has been multiplied by r. For the example S above, we have $S + 3 = \{5,5,6,6,7,8,10\}$ and $3S = \{6,6,9,9,12,15,21\}$.

Rules: In proving each statement you can use the results of previous statements, even if you have not been able to prove them. However, you can only use the results of subsequent statements if you actually prove them.

Problem Group 1:

In this group of problems, you will determine several values of k and n for which $P(k,n)$ is true.

1. **Prove** $P(2,3)$: For every collection, S, of 3 integers there is a subcollection, T, of 2 integers whose sum is divisible by 2.

2. **Lemma**: If T is a collection of k integers which are all equal to each other $\bmod k$ (that is, they all have the same remainder when divided by k), then the sum of the elements in T is divisible by k.

3. **Lemma**: If T is a collection of k integers, then the sum $\bmod k$ of the elements in $T + r$ is equal to the sum $\bmod k$ of the elements in T.

In other words, the sum of the elements in $T + r$ and the sum of the elements in T have the same remainder when divided by k. Consequently, if the sum of the elements in T is divisible by k, then the sum of the elements in $T + r$ is also divisible by k and vice versa.

4. **Prove** $P(4,7)$: For every collection, S, of 7 integers there is a subcollection, T, of 4 integers whose sum is divisible by 4.

5. **Theorem**: If $n = k^2 - k + 1$ then $P(k,n)$ is true.

Problem Group 2:

In this group of problems, you will determine all values of k and n for which $P(k,n)$ is false.

6. **Counterexample to** $P(2,2)$: Find a collection, S, of 2 integers for which there is no subcollection, T, of 2 integers whose sum is divisible by 2.

7. **Counterexample to** $P(4,6)$: Find a collection, S, of 6 integers for which there is no subcollection, T, of 4 integers whose sum is divisible by 4.

8. **Counterexample to** $P(k,2k-2)$: Find a collection, S, of $2k-2$ integers for which there is no subcollection, T, of k integers whose sum is divisible by k.

9. **Lemma**: If $P(k,n)$ is true for some k and n, then $P(k,m)$ is also true for any $m > n$.

10. **Theorem**: For every k, if $n \le 2k-2$, then $P(k,n)$ is false.

Problem Group 3:

In this group of problems, you will determine all values of k and n for which $P(k,n)$ is true. The strategy is to prove $P(k,2k-1)$ when k is prime and separately when k is a product of two numbers for which the theorem has already been proved. An application of mathematical induction will give the theorem for all k's. Since the second part is easier, we prove that first:

11. **Theorem**: If $P(r,2r-1)$ and $P(s,2s-1)$ are true, then $P(rs,2rs-1)$ is true.

HINT: If S is a collection of $2rs-1$ numbers, construct disjoint subcollections $T_1, T_2, \cdots, T_{2s-1}$ each with r elements whose sum is divisible by r.

We next prove $P(p,2p-1)$ is true for primes p. First a lemma:

12. **Lemma**: Assume p is prime. Let A be a subset of $I = \{0,1,2,\cdots,p-1\}$ with n distinct elements where $1 \le n < p$ and let B be a subset of I with 2 distinct elements. Define $(A+B)\bmod p$ to be the set $\{(a+b)\bmod p \mid a \in A \text{ and } b \in B\}$. Then $(A+B)\bmod p$ is a subset of I with at least $n+1$ distinct elements.

13. **Theorem**: If p is prime, then $P(p,2p-1)$ is true.

HINT: Let $S = \{a_1, a_2, \cdots, a_{2p-1}\}$ be the collection of $2p-1$ numbers written in ascending order: $a_1 \le a_2 \le \cdots \le a_{2p-1} \bmod p$. Explain why you can assume no more than $p-1$ of these are equal. Construct T to contain the number a_1 plus one number from each pair $\{a_2, a_{p+1}\}$, $\cdots$, $\{a_i, a_{i+p-1}\}$, $\cdots$, $\{a_p, a_{2p-1}\}$. You need to show that the numbers can be chosen from each pair so that $\Sigma T = 0 \bmod p$. Use the lemma.

14. **Theorem**: If k is a positive integer ≥ 2, then $P(k,2k-1)$ is true.

15. **Corollary**: If $n \ge 2k-1$, then $P(k,n)$ is true.