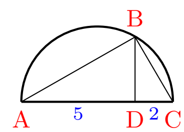
BC Exam Solutions
Texas A&M High School Math Contest
October 24, 2015

All answers must be simplified, and If units are involved, be sure to include them.

1. $p(x) = x^3 + ax^2 + bx + c$ has three roots, λ_i, with $\lambda_1 + \lambda_2 + \lambda_3 = 5$ and $\lambda_1\lambda_2\lambda_3 = -2$. Suppose that $p(1) = 3$. What is $p(2)$?

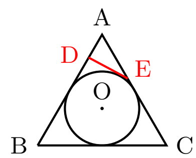
   Solution: We know that $p(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$ from which we see that $a = -5$ and $c = 2$. Thus,
   $$p(1) = 1 - 5 + b + 2 = 3 \implies b = 5\,.$$
   Thus, $p(2) = 8 - 5(4) + 5(2) + 2 = 8 - 20 + 18 + 2 = 0$.

2. Triangle ABC is inscribed in a semicircle with side AC being a diameter of the circle. Line BD is perpendicular to AC with AD=5 and DC = 2. What is the length of BD? See the sketch below.



   Solution: Triangles ABD and BCD are similar. Thus, $\dfrac{5}{BD} = \dfrac{BD}{2}$. and we have $BD = \sqrt{10}$.

3. A circle centered at O is inscribed in the equilateral triangle ABC. Line segment DE is tangent to the circle, is perpendicular to AB, and intersects AB and AC at points D and E respectively. (See the sketch below) If AD = 1 inch, what is the length of one of the triangle's sides?



   Solution: Let G be the point on the circle at which DE is tangent. Draw a straight line from O to F (the point where the circle is tangent to AB) and from O to G. The quadrilateral $GOFD$ is a square, with side length equal to the radius of the circle. Since triangle ABC is equilateral, the radius of the circle, r and the triangle's side length, s satisfy the equation
   $$r = \frac{s}{2\sqrt{3}}\,.$$
   Moreover the point F bisects AB. Thus.
   $$s = 2 \cdot AF = 2\,(r + 1) = 2\left(\frac{s}{2\sqrt{3}} + 1\right)\,.$$
   Solving this equation for s we have
   $$s = \frac{2\sqrt{3}}{\sqrt{3} - 1} = 3 + \sqrt{3}\text{ inches}\,.$$

4. What does $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2015^2}\right)$ equal?

Solution:

$$\begin{aligned}\text{The Product equals} &= \left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\cdots\left(\frac{2015^2-1}{2015^2}\right) = \left(\frac{1\cdot 3}{2^2}\right)\left(\frac{2\cdot 4}{3^2}\right)\cdots\left(\frac{2014\cdot 2016}{2015^2}\right)\\ &= \left(\frac{1}{2}\right)\left(\frac{2016}{2015}\right) = \frac{1008}{2015}\,.\end{aligned}$$

5. $p(x) = x^3 + ax^2 + bx + c$ is odd about the point $x = 2$, and $p(1) = 1$. What does c equal?

Solution: Since p is odd about $x = 2$, $p(2) = 0$, and $p(1) = 1$ implies that $p(3) = -1$. Thus we have the equations:

$$8 + 4a + 2b + c - 0, \quad 1 + a + b + c = 1, \quad 27 + 9a + 3b + c = -1\,.$$

The solution to this system is: $a = -6, b = 10, c = -4$.

6. How many different values of the natural number n are there for which $n^2 - 440$ is a perfect square?

Solution: Suppose k is an integer such that $n^2 - 440 = k^2$. That means

$$n^2 - k^2 = (n-k)(n+k) = 2\cdot 4\cdot 5\cdot 11\,.$$

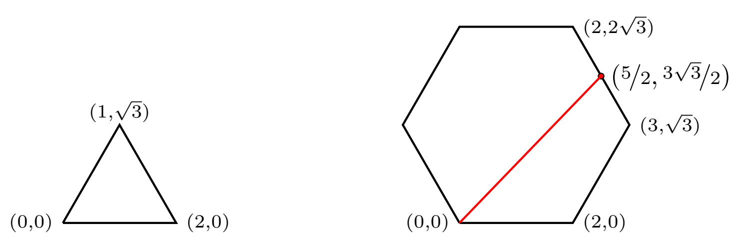
Since $(n-k)(n+k)$ is even, the two terms must have the same parity. In this case the parity must be even. So 4 must divide one of the terms $n-k$ or $n+k$, and 2 must divide the other one. The table below lists the possibilities.

$n-k$	2	$2\cdot 5$	4	$4\cdot 5$
$n+k$	$4\cdot 5\cdot 11$	$4\cdot 11$	$2\cdot 5\cdot 11$	$2\cdot 11$
n	111	27	57	21

Thus, there are 4 possible values of n.

7. A park is in the shape of a regular hexagon and a side length of 2 kilometers. Starting at a corner a woman walks along the perimeter a distance of 5 km. How far is she from her starting point?

Solution: The sketch below shows an equilateral triangle and a regular hexagon, each having side length 2, and the coordinates of their vertices, assuming the left lower corner is at the origin.



The woman's trek ends at the point $(5/2, 3\sqrt{3}/2)$, and the distance from this point to the origin is

$$\text{Distance} = \left(\left(\frac{5}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2\right)^{1/2} = \left(\frac{25+27}{4}\right)^{1/2} = \sqrt{13}\text{ km.}$$

8. The numbers $a_1, a_2, \cdots a_{2015}$ are consecutive terms of an arithmetic sequence, and

$$\sum_{i=1}^{2015} a_i = a_1 + a_2 + \cdots + a_{2015} = 2015\,.$$

Which, if any, of the a_i can be determined, and what are their values?

Solution: The terms of the sequence can be written as:

$$a_i = a_1 + (i-1)k \text{ for } i = 1, \cdots, 2015\,.$$

where k is a fixed constant. Adding these 2015 terms together, we have

$$2015 = \sum_{i=1}^{2015} a_i = \sum_{i=1}^{2015} (a_1 + (i-1)k) = 2015a_1 + k\left(\frac{2014 \cdot 2015}{2}\right)\,.$$

This implies that $a_1 + 1007k = 1$. Since $a_1 + 1007k = a_{1008}$, we have $a_{1008} = 1$.

9. A finite sequence of three digit numbers has the property that the tens and units digits of each term are respectively the hundreds and tens digits of the next term. Moreover, the tens and units digits of the last term are respectively the hundreds and tens digits of the first term. One such sequence is

$$245 \quad 451 \quad 512 \quad 124\,.$$

Let S be the sum of the numbers in the sequence. (S for the above sequence is 1332.) What is the largest prime number that divides S, for all possible S?

Solution: Suppose that

$$x_1x_2x_3 \quad x_2x_3x_4 \quad \cdots \quad x_{n-1}x_nx_1 \quad x_nx_1x_2$$

is one such sequence. If we add the terms, we have

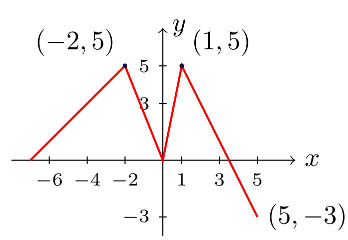
$$S = (x_3 + x_4 + \cdots + x_1 + x_2) + 10\,(x_2 + x_3 + \cdots + x_1) + 100\,(x_1 + x_2 + \cdots + x_n) = 111\,(x_1 + x_2 + \cdots + x_n)\,.$$

Thus, every such S is divisible by $111 = 3 \cdot 37$. Hence the largest prime factor is 37.

10. How many different 15 digit numbers can be made up from the integers 1, 2, and 3, assuming that 1 is not in the first five digits, 2 is not in the second five digits, and 3 is not in the last 5 digits?

Solution: This is essentially the same as asking how many 15 digit numbers can be made up from just 1 and 2. The answer is 2^{15}.

11. The graph of the function $f(x)$ is shown below. How many solutions does the equation $f(f(x)) = 5$ have?



Solution: The equation $f(y) = 5$ has two solutions: $y = -2$ and $y = 1$. The equation $f(x) = -2$ has one solution, which lies between 3 and 5. The equation $f(x) = 1$ has 4 solutions: two positive and two negative. Thus, the equation $f(f(x)) = 5$ has 5 solutions.

12. Triangle ABC has side lengths $\overline{AB} = 5$, $\overline{BC} = 6$, and $\overline{AC} = 7$. Two bugs starting at the same time from A crawl in opposite directions along the sides of the triangle at the same speed. They meet at point D. What is $\overline{BD}$?

Solution: The perimeter of the triangle is 18. Thus, each bug crawls 9 units. D must lie 2 unit from C and 4 units from B. That is, $\overline{BD} = 4$

13. The integers 1 through 9 are written on separate pieces of paper, which are then placed in a hat. A slip is drawn, the number on it noted, and then the slip is returned to the hat. This is done one more time. Which integer is most likely to be the units digit of the sum of the two numbers recorded?

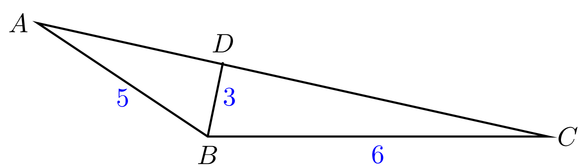
    Solution: The number 0 occurs most often as the units digit. It will occur 9 times, while each of the other digits will occur exactly 8 times.

14. Simplify the expression
$$\frac{bx\left(a^2x^2+2a^2y^2+b^2y^2\right)+ay\left(a^2x^2+2b^2x^2+b^2y^2\right)}{bx+ay}.$$

    Solution:

$$\begin{aligned}
&\frac{bx\left(a^2x^2+2a^2y^2+b^2y^2\right)+ay\left(a^2x^2+2b^2x^2+b^2y^2\right)}{bx+ay} \\
&\qquad = \frac{bx\left(a^2x^2+b^2y^2\right)+ay\left(a^2x^2+b^2y^2\right)+2a^2bxy^2+2ab^2x^2y}{bx+ay} \\
&\qquad = \frac{(bx+ay)\left(a^2x^2+b^2y^2\right)+(bx+ay)\,(2abxy)}{bx+ay} \\
&\qquad = a^2x^2+b^2y^2+2abxy=(ax+by)^2
\end{aligned}$$

15. In the triangle below $\overline{AB}=5, \overline{BC}=6$, and $\overline{BD}=3$. Moreover line $\overline{BD}$ is perpendicular to line $\overline{AC}$. What is the area of triangle ABC?



    Solution: Triangles ABD and BCD are right triangles. Hence,

$$\overline{AD}^2=25-9=16 \implies \overline{AD}=4, \quad \overline{CD}^2=36-9=27 \implies \overline{CD}=\sqrt{27}\,.$$

    Thus,
$$\text{Area of triangle } ABC=\frac{3}{2}\left(4+\sqrt{27}\right)=6+\frac{9}{2}\sqrt{3}$$

16. Find the sum of all positive integers n for which $n-2$ divides n^2+1. Note, negative integers are allowable divisors.

    Solution: Dividing n^2+1 by $n-2$, we have
$$n^2+1=(n-2)(n+2)+5\,.$$

    Thus, $n-2$ divides n^2+1 if and only if $n-2$ divides 5. Thus, $n-2$ must equal one of
$$-5,-1,1,5\,,$$

    which means that $n=-3,1,3$, or 7. The sum of those n's, which are positive is 11.

17. Numbers can be represented in different bases. For example 247 is assumed to be written in base 10 and means $2 \cdot 10^2 + 4 \cdot 10^1 + 7 \cdot 10^0$, while $(247)_8$ (base 8) means $2 \cdot 8^2 + 4 \cdot 8^1 + 7 \cdot 8^0$ or, as normally expressed in base 10, 167. Find the base b such that

$$(43)_{10} = (111)_b \,.$$

Solution: Convert everything to base 10, which means

$$43 = b^2 + b + 1 \implies b^2 + b - 42 = 0 \implies b = \frac{-1 \pm \sqrt{169}}{2} = -7 \text{ or } 6\,.$$

Since $b > 0$, we must have $b = 6$.

18. In the following Sudoku puzzle you are to fill in the missing digits so that each row, column, and small 3×3 squares each contain all of the digit 1 through 9 inclusive. Let a_{ij} denote the digit in the i^{th} row and j^{th} column of the Sudoku puzzle, where a_{11} denoted the entry in the top left hand position, and $a_{1,9}$ denoted the entry in the top right hand position. What is the value of the following sum: $a_{51} + a_{59}$?

9	2					1		
			8		4		2	
4	7	5		9	2		3	
	5	7	4	6	9	8		
	9	4				5	6	
		6	5	2	1	9	7	
	4		6	7		3	9	1
	6		9		3			
		9					8	6

Solution: The entries a_{54}, a_{55}, and a_{56} must consist of the numbers 3, 7, and 8, which means the entries a_{51} and a_{59} must be 1 and 2. Thus, their sum is 3.