2015 Best Student Exam Open
Texas A&M High School Students Contest
October 24, 2015

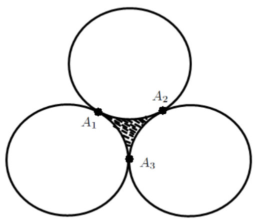
1. It is known that $x + y = 5$ and $x + y + x^2y + xy^2 = 24$. Find $x^3 + y^3$.

2. What is the minimal positive integer such that dividing it by 2 one gets a square of an integer, dividing it by 3 one gets a cube of an integer, and dividing it by 5 one gets the fifth power of an integer. Write you answer in terms of the prime decomposition of this number.

3. α and β are two angles from the interval $[0, \frac{\pi}{2})$ such that the following two relations hold
$$\begin{cases} 3\sin^2\alpha + 2\sin^2\beta = 1, \\ 3\sin(2\alpha) - 2\sin(2\beta) = 0. \end{cases}$$
Find $\alpha + 2\beta$.

4. In a Chemistry class of 25 students the teacher randomly chooses two students to assist in an experiment. The probability that both chosen students are boys is equal to $\frac{3}{25}$. How many girls are in the Chemistry class?

5. Three circles of radius 1 are tangent one to each other in points A_1, A_2, A_3. Find the area of the curvilinear triangle with sides $\smile{A_1A_2}$, $\smile{A_2A_3}$, and $\smile{A_3A_1}$ being the shorter arc on the corresponding circle, i.e. the area of the shape shaded in the figure.



6. 40 people vote for 4 candidates. Each voter can choose only one candidate, and in the voting results only the number of votes each candidate receives count. What is the number of all possible voting results? You may express your answer in terms of binomial coefficients.

7. $f(x)$ is defined for all real x, except $x = 1$, and satisfies the equality
$(x-1)f\left(\frac{x+1}{x-1}\right) = 2x - 3f(x)$. Find $f(-1)$.

8. Given two parallel straight lines, mark 8 points on one line and 9 points on the other line. Draw the segments between each pair of points lying in two different lines. Assume that if three segments intersect in one point, then that point is an endpoint. What is the total number of points of intersection of all drawn segments that are not the endpoints of these segments? You may express your answer in terms of binomial coefficients.

9. Thirteen boys and d girls participate in math contest. The total number of points they earn is $d^2 + 10d + 17$. It is known that all students get the same integer number of points. Find the largest possible number of participants in this contest.

10. Let $\frac{p}{q}$ be the irreducible fraction (the fraction in lowest terms) such that $\frac{p}{q} = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\dots\left(1 - \frac{1}{225}\right)$. Find $\frac{p}{q}$.

11. Assume that $f(x) = x^2 + 10x + 20$. Find the largest real solution of the equation $f(f(f(f(x)))) = 0$.

12. Consider a triangle C_1C_2O with $\angle C_1OC_2 = \frac{\pi}{6}$. Let C_2C_3 be the bisector in the triangle C_1C_2O, then C_3C_4 be the bisector in the triangle C_2C_3O, and, generally, $C_{n+1}C_{n+2}O$ be the bisector in the triangle $C_nC_{n+1}O$ for any $n \geq 1$. Let $\gamma_n = \angle C_{n+1}C_nO$. Find the limit of γ_n as $n \to +\infty$.

13. Several families participated in a Christmas party. Each family consisted of mother, father, and at least one but not more then 10 children. Santa Claus chose a mother, a father, and one child from three different families for a sleigh ride. It turned out that he had exactly 3630 ways to choose such triples. How many children were in the party?

14. Four ants simultaneously stand on the four vertices of a regular tetrahedron, with each ant at a different vertex. Simultaneously and independently, each ant moves from its vertex to one of the three adjacent vertices, each with equal probability. What is the probability that no two ants arrive at the same vertex?

15. Find the real number x such that there exists exactly one non-integer number among the four numbers $a = x - \sqrt{2}$, $b = x - \frac{1}{x}$, $c = x + \frac{1}{x}$, and $d = x^2 + 2\sqrt{2}$.

16. Assume that $P_1(x), P_2(x), \dots, P_6(x)$ are polynomials such that the sum of coefficients of the polynomial $P_i(x)$ is equal to i for each integer i between 1 and 6. What is the sum of the coefficients of the polynomial $F(x) = P_1(x)P_2(x)\dots P_6(x)$, i.e. of the polynomial obtained by taking the product of all polynomials $P_1(x), P_2(x), \dots, P_6(x)$?

17. Which is bigger π^e or e^π?

18. There are 16 students in class. Every month the teacher divides the class into two groups of 8 students each. After n months every two students were in different groups during at least one month. What is the minimal possible n?

19. Evaluate the following expression:

$$2016 \int_0^\pi |\sin(2015x)|\, dx - 2015 \int_0^\pi |\sin(2016x)|\, dx.$$

20. It is known that the number $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}$ is a rational number. Find this number (in lowest terms).