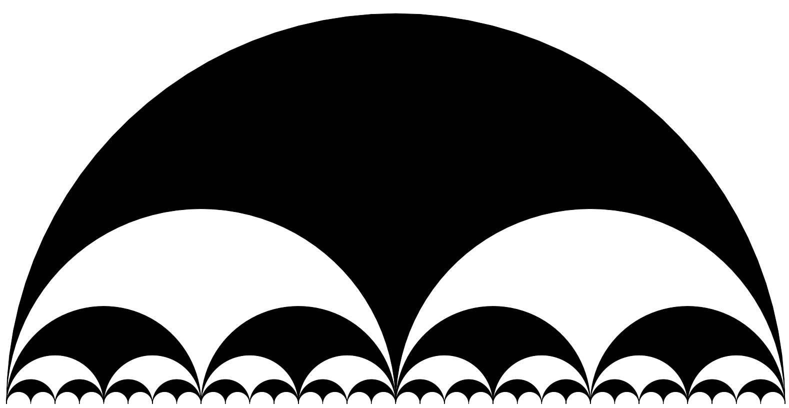
CD EXAM
Texas A&M High School Math Contest
Oct 24, 2015

1. Consider an equilateral triangle ABC with sides of length 1. Extend AB from A to A' so that A lies between A' and B, and the distance from A' to A is x. Similarly extend AC to C' and CB to B'. Triangle $A'B'C'$ will also be equilateral. Find the smallest positive integer value for x so that the sides of $A'B'C'$ have integer lengths.

2. Find
$$\sqrt{2^1 + \sqrt{2^2 + \sqrt{2^4 + \sqrt{2^8 + \cdots}}}}$$
and give the answer in the form $(a+\sqrt{b})/\sqrt{c}$ where a, b, and c are positive integers.

3. Find the pair of positive integers a and b with smallest sum such that for all positive integers k,
$$10^{2(ak+b)} + 10^{ak+b} + 9$$
is divisible by 7.

4. A half circle of radius 1 has two half circles inscribed along its base, each of radius 1/2. These have further half-circles inscribed, the pattern continuing to all depths.



What is the sum of the areas of the black-colored portions?

5. A dartboard of radius r has zones bounded by circles of radius $r/4$, $r/2$, and $3r/4$, by the x and y axes, and by the lines $y = \pm x$. What is the farthest distance between two points in a single zone?

6. A polynomial $p(z)$ is called *suitable* if it has the form

$$\begin{aligned} p(z) &= (z - w_1)(z - w_2)(z - w_3)(z - w_4) \\ &= (z - u_1 - iv_1)(z - u_2 - iv_2)(z - u_3 - iv_3)(z - u_4 - iv_4) \end{aligned}$$

where each of the u_j's and v_j's is an integer. Find a suitable polynomial $p(z)$ such that $p(0) = 4$, $p(1) = 5$, and for all z,

$$(z^2 + 2z + 2)p(z - 2) = (z^2 - 6z + 10)p(z).$$

7. For which values of c does the system of equations

$$xy = 2/c, \qquad x^2 + y^2 = c$$

have four distinct real solutions?

8. A man has \$ 1.70 in nickles, dimes, and quarters. He lists how many of each he has, producing a list of the form (n, d, q). (It might read $(4, 0, 6)$, say, or $(2, 1, 6)$, or $(0, 17, 0)$.) How many possibilities are there for this list?

9. A three dimensional chess board has 512 cubical 'squares'. A queen, on this board, can move in any direction a regular queen could move, in any of the three 8 by 8 planes that include here square. She can also move along any of the long diagonals through her position. What is the maximum number of squares such a queen can get to in one move from any particular position on the board?

10. Find $(\log_2 3)(\log_9 16)$.

11. Find the sum of the coordinates of all integer points strictly inside the triangle with vertices $(0, 0)$, $(20, 8)$, and $(21, 9)$.

12. An equilateral triangle with vertices $(-1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, \sqrt{3})$ is rotated around the z axis, forming a cone. A sphere is inscribed in the cone, tangent at a point on the x-y plane and tangent to the curved surface of the cone along a circle. A plane parallel to the x-y plane is also tangent to the sphere at its top. This cuts off a little cone at the top of the big cone. Find the ratio of the volume of the little cone to the volume of the big cone.

13. Given that $x^2 - y^2 = 2$ and $x^3 - y^3 = 3$, find $x + y - (xy/(x + y))$.

14. Let

$$S = \sum_{k=1}^{2015} \frac{1}{k(k+1)(k+2)} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{2015 \cdot 2016 \cdot 2017}.$$

The decimal expansion of S has the form $a.bcdefghijkl\cdots$. Find a, b, c, and d.

15. Find the largest integer k such that 2^k divides $100!/(50!)^2$.

16. Find all ordered triples of integers (a, b, c) so that $|a+b| + |b+c| = 1$ and $|a+b| + |a+c| = 3$.

17. Find the least prime p that divides $10^{\left(10^{10}\right)} + 10^{10} + 10 - 1$.