DE Exam
Texas A&M High School Math Contest
October 24, 2015

Answers should include units when appropriate.

1. Two ferry boats ply back and forth across a river with constant speeds, turning at the banks without loss of time. They leave opposite shores at the same instant, meet for the first time 700 feet from one shore, continue on their way to the banks, return and meet for the second time 400 feet from the opposite shore. Determine the width of the river.

2. Three men, A, B, and C, working together, do a job in 6 hours less time than A alone, in 1 hour less time than B alone, and in one-half the time needed by C when working alone. What is the time needed by A and B, working together to do the job.

3. Find the smallest integer n such that $\sqrt{n} - \sqrt{n-1} < 0.01$.

4. Find all possible values of the expression

$$\frac{|x+y|}{|x|+|y|} + \frac{|y+z|}{|y|+|z|} + \frac{|z+x|}{|z|+|x|},$$

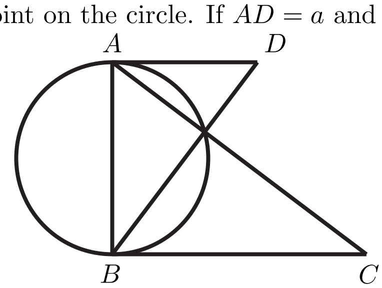
where x, y, z are arbitrary non-zero real numbers.

5. Solve the system

$$\begin{cases} x &= x^2 + y^2 \\ y &= 2xy \end{cases}$$

6. If $\sin x + \cos x = 1/5$ and $0 \le x < \pi$, find $\tan x$.

7. Let AB be a diameter of a circle. Tangents AD and BC are drawn so that AC and BD intersect in a point on the circle. If $AD = a$ and $BC = b$, find the diameter of the circle.



8. Let a, b be positive real numbers. Find the values of m for which the equation

$$|x-a| + |x-b| + |x+a| + |x+b| = m(a+b)$$

has at least one real solution.

9. If $\cos 2\alpha = m$, find $\sin^6 \alpha + \cos^6 \alpha$.

10. Find the remainder obtained by dividing x^{2015} by $x^2 - 3x + 2$.

11. Solve the system

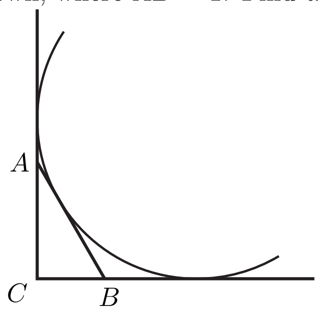
$$\begin{cases} \log_y x + \log_x y &= 5/2 \\ xy &= 27 \end{cases}$$

12. Solve the equation

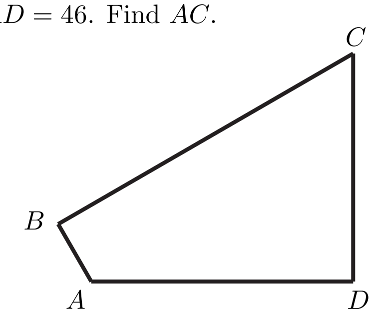
$$\left(\sqrt[3]{0.5} + \sqrt[3]{4}\right)^x = 13.5.$$

13. Find all pairs (x, y) of real numbers such that $x^2 + 2x\sin(xy) + 1 = 0$.

14. A circle is tangent to the coordinate axes and to the hypotenuse of the $30° - 60° - 90°$ triangle ABC as shown, where $AB = 1$. Find the radius of the circle.



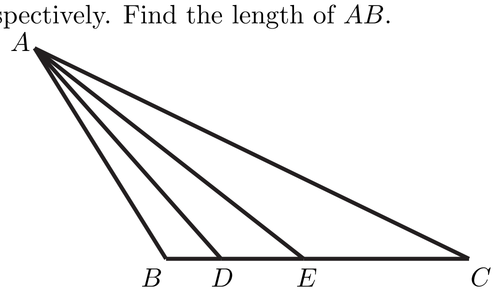
15. In a quadrilateral $ABCD$, it is given that $\angle A = 120°$, angles B and D are right angles, $AB = 13$, and $AD = 46$. Find AC.



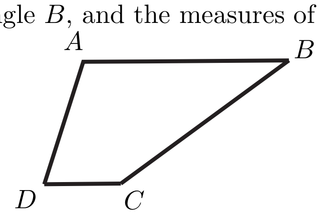
16. Find all values of c such that the polynomials $cx^3 - x^2 - x - (c+1)$ and $cx^2 - x - (c+1)$ have a common root.

17. Find all x such that $0 \le x < 2\pi$ and $\sin 3x = 2\sin x$.

18. In a triangle ABC, AD and AE trisect $\angle BAC$. The lengths of BD, DE, and EC are 2, 3, and 6, respectively. Find the length of AB.



19. In the quadrilateral $ABCD$, segments AB and CD are parallel, the measure of angle D is twice that of angle B, and the measures of segments AD and CD are a and b respectively. Find the measure of AB.



20. Suppose that the function $f(n)$ satisfies $f(x) + f(y) = f(x+y) - xy - 1$ for every pair x, y of real numbers. If $f(1) = 1$, find all integers n such that $f(n) = n$.

21. If $P(x)$ denotes a polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$, determine $P(n+1)$.