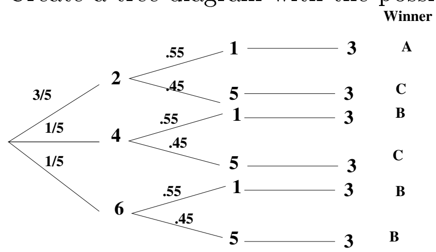
Solutions to EF Exam

Texas A&M High School Math Contest
24 October, 2015

1. Create a tree diagram with the possibilities:



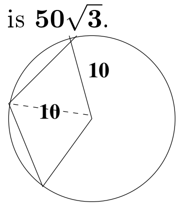
$P(A \text{ wins}) = 3/5 * 0.55 = 0.33$; $P(B \text{ wins}) = 1/5 * 0.55 + 1/5 * 0.55 + 1/5 * 0.45 = 0.31$; $P(C \text{ wins}) = 3/5 * 0.45 + 1/5 * 0.45 = 0.36$, so the most likely winner is **C**.

2. Let d_1 be the distance the first freshman had traveled and d_2 be the distance the second freshman had traveled when they met. If x is the speed of the first and y the speed of the second, we have $\dfrac{d_1}{x} = 4\dfrac{1}{12} = \dfrac{49}{12}$ and $\dfrac{d_2}{y} = 2\dfrac{1}{12} = \dfrac{25}{12}$, for a total distance of $\dfrac{49x + 25y}{12}$. Since they arrived at the same time, $\dfrac{49 + 25\frac{y}{x}}{12} = \dfrac{49\dfrac{x}{y} + 25}{12} + 2$. Multiplying by 12 and simplifying we obtain $\dfrac{25y}{x} = \dfrac{49x}{y}$, so $y = \dfrac{7}{5}x$ and the first freshman's total time is $\dfrac{49 + 35}{12} = 7$ hours. **7:00pm**

3. Factoring the left hand side yields $(x(x-8)(x+8))\,(x(x+8))\,(9(x-8)) = 9x^2(x-8)^2(x+8)^2$, which is always greater than or equal to 0. $(-\infty, \infty)$

4. The fraction is equivalent to $\dfrac{(76.56)(84.91) - (76.56)(35.72)}{(49.19)(0.634) + (0.566)(49.19)} = \dfrac{(76.56)(49.19)}{(49.19)(1.2)} = \dfrac{76.56}{1.2} = \mathbf{63.8}$.

5. From the figure below, one diagonal of the rhombus is also a radius of the circle, thus forming 2 equilateral triangles, each of whose area is $\dfrac{(10)^2\sqrt{3}}{4} = 25\sqrt{3}$. Therefore, the area of the rhombus is $\mathbf{50\sqrt{3}}$.



6. Let x be the lengths of the sides of $\triangle PQR$ and s be the lengths of the sides of $\triangle ABC$. Draw altitude $\overline{AD} \perp \overline{BC}$. Then $\triangle ADC \sim \triangle QPC$, meaning $PC = \dfrac{x}{\sqrt{3}}$. Likewise, draw altitude $\overline{CE} \perp \overline{AB}$. Then $\triangle CEB \sim \triangle PRB$, meaning $BP = \dfrac{2x}{\sqrt{3}}$. Therefore, $s = \dfrac{x}{\sqrt{3}} + \dfrac{2x}{\sqrt{3}} = \sqrt{3}\,x$, so the ratio of the areas is $\dfrac{x^2}{s^2} = \mathbf{\dfrac{1}{3}}$.

7. Let a and $a+5$ be the roots. Then $a + (a+5) = -\dfrac{m}{3}$ and $a(a+5) = -4$. From the second equation, $a = -4$ or $a = -1$, meaning $m = \pm 9$, so the product is $\mathbf{-81}$.

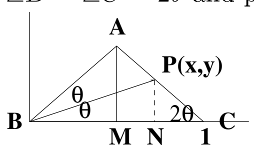
8. Multiply both equations by $\dfrac{1}{3}$ and add the results, giving us $x - y - z = \mathbf{5}$.

9. $\dfrac{1-\tan\left(\frac{\pi}{12}\right)}{1+\tan\left(\frac{\pi}{12}\right)} = \dfrac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{\pi}{12}\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\dfrac{\pi}{4}-\dfrac{\pi}{12}\right) = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{3}}$.

10. $a^2e^{-ax} - b^2e^{-bx} = b^2e^{-bx}\left(\left(\dfrac{a}{b}\right)^2 e^{-(a-b)x} - 1\right)$. Substituting x in the parentheses yields $b^2e^{-bx}\left(\left(\dfrac{a}{b}\right)^2 e^{-2(\ln a - \ln b)}\right.$ - $b^2e^{-bx}\left(\left(\dfrac{a}{b}\right)^2\left(\dfrac{a}{b}\right)^{-2} - 1\right) = \mathbf{0}$.

11. Differentiate with respect to t to obtain $\dfrac{dy}{dt} = (2x+2)\dfrac{dx}{dt}$. We want $\dfrac{dy}{dt} = \dfrac{dx}{dt}$, so $\dfrac{dx}{dt} = (2x+2)\dfrac{dx}{dt}$ or $0 = (2x+1)\dfrac{dx}{dt}$. Therefore, $x = -\dfrac{1}{2}$, meaning $y = -\dfrac{3}{4}$. $\left(-\dfrac{\mathbf{1}}{\mathbf{2}}, -\dfrac{\mathbf{3}}{\mathbf{4}}\right)$.

12. Draw a coordinate system with the origin at point B and label as shown below, such that $\angle B = \angle C = 2\theta$ and point P has coordinates (x, y):



From $\triangle CNP$, $\tan(2\theta) = \dfrac{y}{1-x}$, and from $\triangle BNP$, $\tan(\theta) = \dfrac{y}{x}$. Applying the double-angle tangent formula yields $\dfrac{y}{1-x} = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)} = \dfrac{2\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)^2} = \dfrac{2xy}{x^2-y^2}$. Since $y \neq 0$, cancel and cross-multiply to obtain $x^2 - y^2 = 2x - 2x^2$, or $y^2 = 3x^2 - 2x$. As $AM \to 0$, $PN = y \to 0$, so we must have $x(3x-2) \to 0$. This occurs when $x \to 0$ (not possible since by construction x must be $\geq \dfrac{1}{2}$ or $x \to \dfrac{2}{3}$. So $BP \to \dfrac{\mathbf{2}}{\mathbf{3}}$.
(Alternately, since the angle bisector of a triangle divides the opposite side proportionally to the adjacent sides of the triangle, we have $\dfrac{AB}{BC} = \dfrac{AP}{PC}$. As point A approaches point M, $\dfrac{AB}{BC} \to \dfrac{1}{2}$, meaning $\dfrac{AP}{PC} \to \dfrac{1}{2}$ So $BP \to \dfrac{1}{2} + \dfrac{1}{3}\cdot\dfrac{1}{2} = \dfrac{\mathbf{2}}{\mathbf{3}}$.)

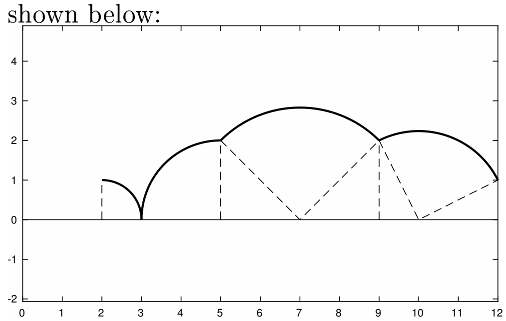
13. For $t \leq 0$, $f(t) = 3t+3$, so $g(-2) = \displaystyle\int_0^{-2}(3t+2)\,dt = -\int_{-2}^{0}(3t+2)\,dt = -\left(\frac{3}{2}t^2 + 2t\right)\Big|_{-2}^{0}$ $= -(0-2) = 2$. $g'(-2) = f(-2) - 1 = -4$, and $g''(-2)$ represents the slope of the line, which is 3. So the product is $-\mathbf{24}$.

14. We want $\displaystyle\lim_{x\to\infty} \ln\left(\left(\frac{x+a}{x-a}\right)^x\right) = 1$. $\displaystyle\lim_{x\to\infty} \ln\left(\left(\frac{x+a}{x-a}\right)^x\right) = \lim_{x\to a}\frac{\ln(x+a)-\ln(x-a)}{\frac{1}{x}}$. Apply L'Hospital's Rule to obtain $\displaystyle\lim_{x\to\infty}\frac{\frac{1}{x+a}-\frac{1}{x-a}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{2ax^2}{x^2-a^2} = 2a$. Hence, $2a = 1$, and $a = \dfrac{\mathbf{1}}{\mathbf{2}}$.

(Alternately, we know $\left(\dfrac{x+a}{x-a}\right)^x = \left(1+\dfrac{2a}{x-a}\right)^x = \left(1+\dfrac{2a}{x-a}\right)^{\left(\frac{x-a}{2a}\right)\left(\frac{2ax}{x-a}\right)}$ which approaches e^{2a} and hence leads to our desired result.)

15. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2u}\,dx$. Substituting and changing the boundaries yields $\displaystyle\int_0^{\pi/2} 2u\cos(u)\,du$. Integrate by Parts to obtain $2u\sin(u) + 2\cos(u)|_0^{\pi/2} = \pi - \mathbf{2}$

16. The point travels in circular arcs from (2, 1) to (3, 0) to (5, 2) to (9, 2) and finally to (12, 1) as shown below:



The area consists of two 2×2 right triangles of area 2, two 1×2 right triangles of area 1, and four quarter-circles of radii 1, 2, $\sqrt{8}$, and $\sqrt{5}$ respectively. Therefore, the area is $6 + \frac{1}{4}(\pi + 4\pi + 8\pi + 5\pi) = \mathbf{6 + \frac{9\pi}{2}}$.

17. Differentiate $f^{(n-1)}(x) = \dfrac{P_{n-1}(x)}{(x^{2015}-1)^n}$ to obtain

$f^{(n)}(x) = \dfrac{(x^{2015}-1)^n P'_{n-1}(x) - P_{n-1}(x)(n)(x^{2015}-1)^{n-1}(2015x^{2014})}{(x^{2015}-1)^{2n}}$

$= \dfrac{(x^{2015}-1)P'_{n-1}(x) - 2015P_{n-1}(x)nx^{2014}}{(x^{2014}-1)^{n+1}}$. Then $P_n(x) = (x^{2015}-1)P'_{n-1}(x) - 2015P_{n-1}(x)nx^{2014}$ and $P_n(1) = 0 - 2015nP_{n-1}(1)$. Since $\dfrac{d}{dx}\left(\dfrac{1}{(x^{2015}-1)}\right) = \dfrac{-2015x^{2014}}{(x^{2015}-1)^2}$, it can be shown inductively that, if $\dfrac{d^n}{dx^n}\left(\dfrac{1}{x^{2015}-1}\right) = \dfrac{P_n(x)}{(x^{2015}-1)^{n+1}}$, that $P_n(x) = (-2015)^n n!$ Therefore, the 2015th derivative is $(\mathbf{-2015})^{\mathbf{2015}}(\mathbf{2015})!$

18. Differentiate both sides of the equation to obtain $2f(x)f'(x) = (f(x))^2 + (f'(x))^2$. This is true for all x provided $(f(x) - f'(x))^2 = 0$, or $f(x) = f'(x)$, meaning $f(x) = Ae^x$. To find a value of A which satisfies the original equation for all x, note that when $x = 0$, $(f(0))^2 = A^2 = 0 + 2015$, or $A = \pm\sqrt{2015}$. Since $f(x) > 0$ for all x, $\mathbf{f(x) = \sqrt{2015}e^x}$.

19. Let $u = 6 - x$. Then $u + 6 = 12 - x$ and $12 - u = x + 6$. Let I be the value of the original integral. Substituting and changing boundaries yields $I = \displaystyle\int_1^5 \frac{(12-u)\sin(u+6)}{(12-u)\sin(u+6) + (u+6)\sin(12-u)}\,du$.
Therefore, $2I = \displaystyle\int_1^5 \frac{(x+6)\sin(12-x) + (12-x)\sin(x+6)}{(12-x)\sin(x+6) + (x+6)\sin(12-x)}\,dx = \int_1^5 1\,dx = 4$, so $I = \mathbf{2}$.