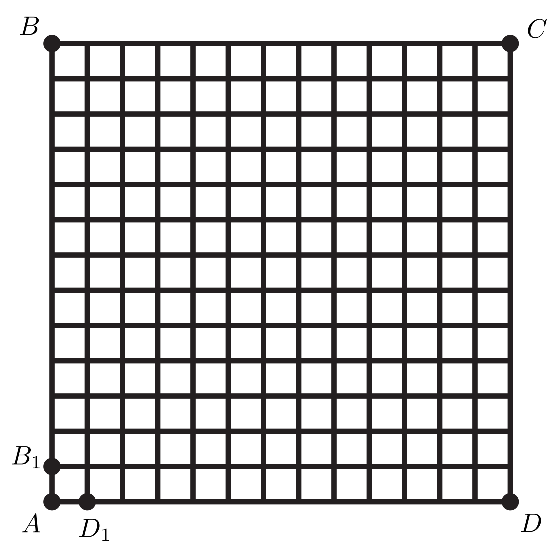
2016 Power Team Exam

Problem 1. A square is divided into 10,000 equal squares. Each of the little squares is colored red or blue in such a way that in every row and in every column there are exactly 50 red and 50 blue squares. The centers of every two adjacent red squares (i.e., squares sharing a common side) are connected by a red segment, and the centers of every two adjacent blue squares are connected by a blue segment. Prove that the number of red segments is equal to the number of blue segments.

Problem 2. Consider a rectangle $ABCD$ on the square grid, as it is shown on the figure (the number of squares and the lengths of the sides can be arbitrary, not necessarily as on the figure). Let B_1 and D_1 be the vertices of the grid adjacent to A on the sides AB and AD, respectively.



A bee can travel only along the sides of the squares of the grid (*edges*).

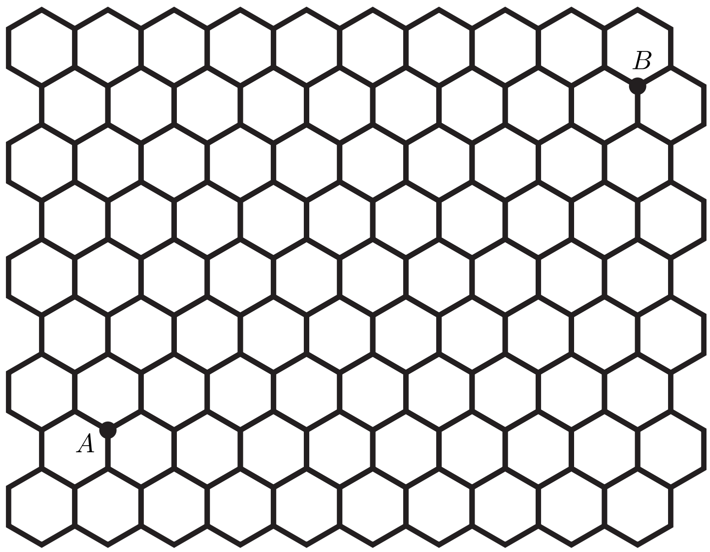
(a) Prove that the number of shortest paths for the bee from A to C is equal to the sum of the number of shortest paths from B_1 to C and the number of shortest paths from D_1 to C.

(b) Prove that the number of shortest paths for the bee from B_1 to C divided by the number of shortest paths from D_1 to C is equal to $\frac{|AB|}{|AD|}$, where $|XY|$ denotes the length of the segment XY.

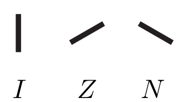
Problem 3. Consider now the infinite square grid in the plane. Prove that every closed path (i.e., starting and ending in the same vertex) of the bee contains an even number of edges.

Problem 4. Prove that if the bee makes a $90°$ turn (left or right) at every vertex on its way, then it can come back to the beginning of its path only after passing a number of edges divisible by 4.

In the subsequent problems the bee can travel along the sides of the hexagons in the hexagonal grid, shown on the next figure.



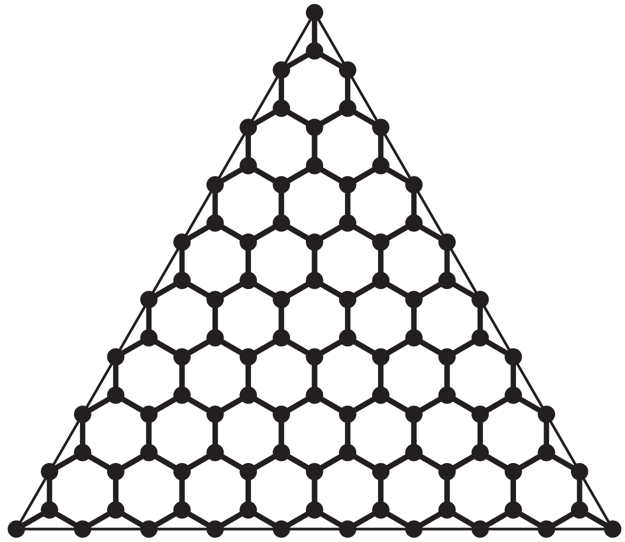
Note that there are three possible directions of the edges, which we will denote by the letters I, Z, and N, as it is shown below.



Problem 5. Prove that every closed path in the hexagonal grid contains an even number of edges.

Problem 6. Let γ be a path of the bee, and let $e_1, e_2, \ldots, e_n$ be the sequence of the edges traversed by it. Denote by I_0, Z_0, N_0 the numbers of the edges of the types I, Z, N, respectively, among the edges $e_2, e_4, e_6, \ldots$ on the even places, and by I_1, Z_1, N_1 the numbers of the edges of the types I, Z, N, respectively, among the edges $e_1, e_3, e_5, \ldots$ on the odd places. Find a necessary and sufficient condition in terms of the numbers $I_0, I_1, Z_0, Z_1, N_0, N_1$ for the path γ to be closed.

Problem 7. Consider two vertices of the hexagonal grid A and B. Suppose that the shortest path for the bee from A to B has even length $2n$. Prove that there exists a type I, Z, or N such that exactly n edges in the shortest path are of this type.



Problem 8. Consider the part of the grid inside an equilateral triangle, as it is shown on the figure above. Suppose that the number of vertices of the grid on one side of the triangle is k. What is the maximal number of vertices in this triangle the bee can visit along one path, if it is allowed to visit each vertex at most once (and, as before, can travel only along the sides of the hexagons).

Problem 9. We say that a vertex of the hexagonal grid is n-reachable from a vertex A if it can be reached by the bee after crawling through n or less edges. How many vertices are n-reachable from A?