Problem 1. Consider all segments connecting the centers of adjacent squares. They are of three types: blue, red, and the segments connecting squares of different color. Let B, R, M be the numbers of the segments of each type, respectively.

The left ends of the horizontal segments belong to 99 columns (all but the rightmost column) of the square, hence they belong to equal numbers of red and blue squares. Similarly, the right ends of the horizontal segments belong to 99 columns, and hence they belong to equal numbers of red and blue squares. The same argument is valid for the vertical edges. It follows that the number of ends of segments that belong to red squares is equal to the number of ends of segments that belong to blue squares. The first number is equal to $2R+M$, the second number is $2B+M$. It follows that $R = B$.

Problem 2. (a) The first step from A will be either to B_1, or to D_1. In the first case the bee has to continue along a shortest path from B_1 to C, in the second case it must continue along a shortest path from D_1 to C. The statement of the problem follows.

(b) Denote by $P(m, n)$ the number of shortest paths connecting opposite vertices of a rectangle of size $m \times n$. We have show in (a) that

$$P(m, n) = P(m-1, n) + P(m, n-1).$$

We can also naturally define $P(m, 0) = 1$ and $P(0, n) = 1$, which agrees with the recurrent formula.

We have to prove that $\frac{P(m-1,n)}{P(m,n-1)} = \frac{m}{n}$ for all $m, n \geq 1$. Let us prove it by induction. It is true for every pair (m, n), where m or n is equal to 1, since we have $P(0, n) = P(n, 0) = 1$, and $P(1, n-1) = n$ and $P(m-1, 1) = m$. This will be the base of our induction. For the inductive step, note that

$$\frac{P(m-1,n)}{P(m,n-1)} = \frac{P(m-2,n) + P(m-1,n-1)}{P(m-1,n-1), +P(m,n-2)} =$$

$$\frac{\frac{P(m-2,n)}{P(m-1,n-1)} + 1}{1 + \frac{P(m,n-2)}{P(m-1,n-1)}} = \frac{\frac{m-1}{n} + 1}{\frac{n-1}{m} + 1} = \frac{m(m-1+n)}{n(n-1+m)} = \frac{m}{n}.$$

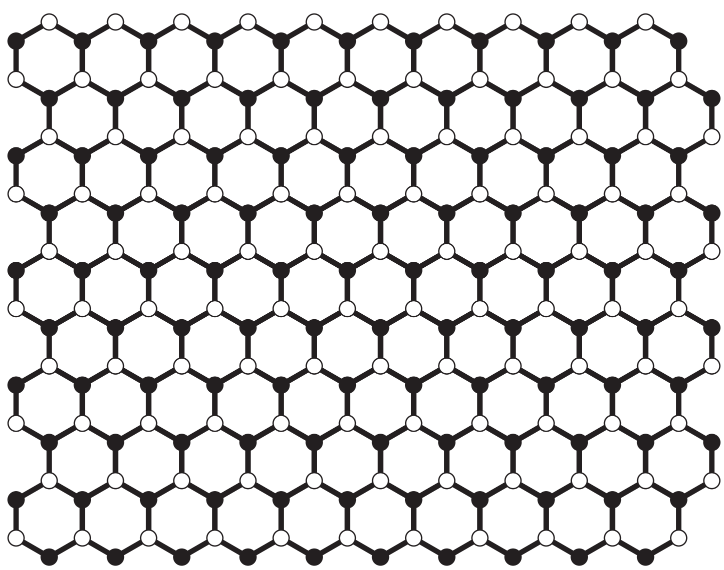
Remark. The recurrent formula in part (a) shows that $P(m, n) = \binom{m+n}{n} = \binom{m+n}{m}$, which is also easy to prove directly (for example, noting that every path is determined by the decision which among $m + n$ segments on the path are the n horizontal ones). Then (b) can be proved using the formula $\binom{m+n}{n} = \frac{(m+n)!}{m!n!}$, or, conversely, (b) can be used to prove this formula.

Problem 3. Let us introduce a coordinate system such that the vertices of the grid are exactly the points with integer coordinates. Then after the bee crawls along one edge of its path, its x or y coordinate is changed by 1, while the other coordinate is unchanged. It follows that the parity of $x + y$ is changed each time. Thus, if the path is closed, then the number of edges passed by the bee must be even, since the value of $x+y$ at the beginning and at the end of the path must be the same.

An equivalent solution is to color the vertices of the grid in two colors so that any two vertices connected by an edge have opposite color. Then in any path the colors of the vertices will alternate, hence every closed path has even length.

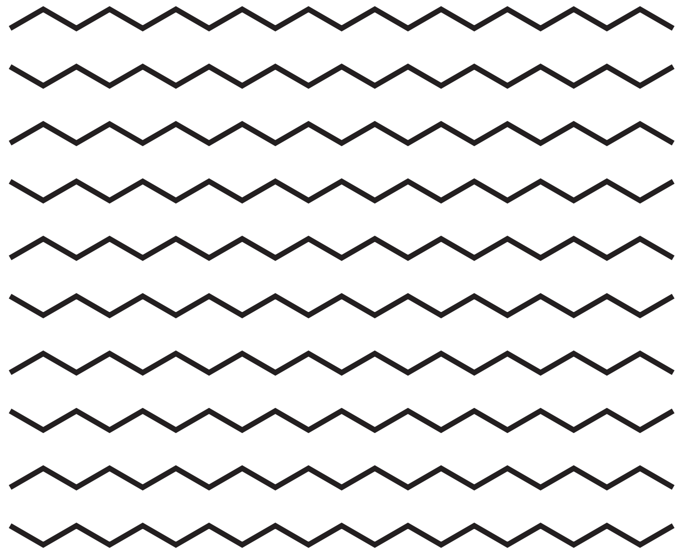
Problem 4. We know that the bee has to pass an even number of edges. Let us group the edges on its path into pairs: first and second, third and fourth, etc.. Then after traversing each pair of edges the bee will change its position from a vertex of a square of the grid to the opposite vertex of the square. In other words, the bee will be traveling along the diagonals of the squares. The diagonals that it can visit will form a square grid, and by Problem 3, the number of diagonals traversed by the bee has to be even. Hence, the number of pairs of edges passed by the bee must be even, i.e., the number of the edges must be divisible by 4.

Problem 5. We can color the vertices of the hexagonal grid in two colors (white and black) so that every two adjacent vertices have different color, see the figure below.



Then the colors of the vertices on the path will alternate, hence the path must have even length.

Problem 6. *Solution 1.* If we remove from the grid the edges of type I, then the grid will be split into a sequence of paths, like in the figure below.



We will call these paths I-*levels*, and order then from bottom to top in the natural way (assuming that the edges of type I are vertical).

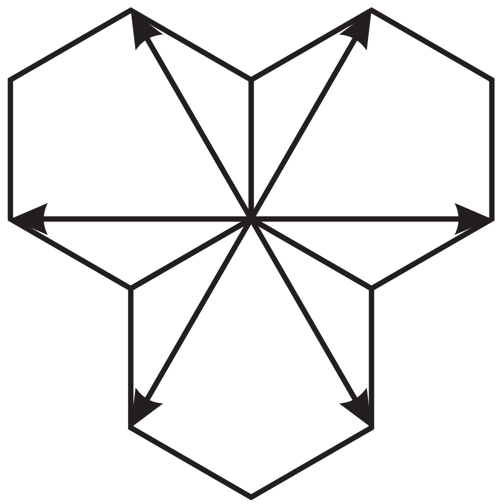
Let us color the vertices of the grid in two colors as in the figure from the solution of Problem 5, and let us assume that the path has started in a white vertex. Then it is easy to see that when the bee passes an edge of type I on an odd place of its path, then the bee raises one level higher. If it passes an edge of type I on an even place, then it descends one level lower. It follows that we must have $I_0 = I_1$ in every closed path. Similar arguments show that $N_0 = N_1$ and $Z_0 = Z_1$ for every closed path.

It is easy to see that intersection of one I-level with one Z-level is one edge of type N. Consequently, if the bee returns to the same I-level, to the same Z-level, and to the same N-level, then it returns to the same vertex. Thus, the condition $I_0 = I_1$, $N_0 = N_1$, and $Z_0 = Z_1$ is both necessary and sufficient.

Solution 2. Consider a path γ, and let $X_1X_2X_3\ldots$ be the sequence of types of the edges in the order they appear on the path, i.e., so that X_i is the type of the edge e_i. Note that since every vertex is an end of exactly one edge of each type, this sequence together with the initial vertex uniquely determine the path.

According to Problem 5, every closed path must have even length. Let us group the types X_i into pairs: $(X_1X_2)(X_3X_4)(X_5X_6)\ldots$. Let us also color the vertices of the grid, as in Problem 4, and assume that γ starts in a white vertex. Then it will be in a white vertex after completing each pair of steps. Consider a coordinate system such that the type I edges are parallel to the y-axis, and the width of the hexagons (i.e., length of their shorter diagonals) is equal to 1. Below is then a table showing how the x-coordinate of the bee is changed after it completes one pair of edges, depending on their type, see the figure below.

II	IZ	IN	ZI	ZZ	ZN	NI	NZ	NN
$+0$	$+1/2$	$-1/2$	$-1/2$	$+0$	-1	$+1/2$	$+1$	0

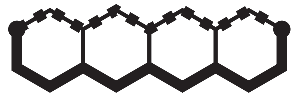


We can interpret these rules in the following way. Type Z on the first place has value $-1/2$, type Z on the second place has value $1/2$, type N on the first place has value $1/2$, type N on the second place has value $-1/2$, type I has zero value on both places. Then it is easy to check using the above table that in each case the sum of values is equal to the change of the x-coordinate. It follows that the net change of the x-coordinate of the bee along the path γ is equal to $\frac{1}{2}(Z_0 - Z_1 - N_0 + N_1)$. Consequently, it is zero if and only if $Z_0 - Z_1 - N_0 + N_1 = 0$, or $Z_0 - Z_1 = N_0 - N_1$.

Consider now a coordinate system such that the edges of type Z are parallel to the y-axis. Then the net change of the x-coordinate of the bee in this coordinate system is equal to zero if and only if $Z_0 - Z_1 = I_0 - I_1$.

The path is closed if and only if both conditions are satisfied. It follows that the path is closed if and only if the differences $I_0 - I_1$, $Z_0 - Z_1$, and $N_0 - N_1$ are equal and the length of the path is even. Note that the sum of the differences $(I_0 - I_1) + (Z_0 - Z_1) + (N_0 - N_1)$ is equal to 0 if the length of the path γ is even, and to -1 otherwise. If the differences are equal then the latter case is impossible. It follows that if they are equal, then they are all equal to 0. Consequently, γ is closed if and only if $I_0 = I_1$, $Z_0 = Z_1$, and $N_0 = N_1$.

Problem 7. Consider two edges of type I in the path such that all the edges between them are of type N or Z. Then the types of the edges between the I-edges will alternate. It is easy to see that if their number is even, then the path can be shortened: for example, the path $ZNZNZNZN$ connects the same vertices as the path $INZNZNZNZI$, see the figure below.



It follows that the edges of type I are either only on even places, or only on odd places. The same is true for the other two types Z and N. But there are only three possible types, so one of the types appears on the places of one parity, and the other two types appear on the places of the other parity. It follows that one of the types appears exactly half of time.

Problem 8. Color the vertices of the grid as in Problem 5. Then the triangle will contain $\frac{k(k+1)}{2}$ vertices of one color and $\frac{k(k-1)}{2}$ vertices of the other color. The colors of the vertices on the path will alternate, hence the path can contain at most $2 \cdot \frac{k(k-1)}{2} + 1 = k^2 - k + 1$ vertices. In other words, the number of missing vertices will be $\frac{k(k-1)}{2} + \frac{k(k+1)}{2} - (k^2 - k + 1) = k - 1$. The path shown in the figure below will have the required number of vertices.

Problem 9. We know from the solution of Problem 7, that in every shortest path on the grid there is one type I, Z, or N and one parity, such that all edges of that parity along the path are of that type, while all the edges of the opposite parity are of the two remaining types. Consider a path γ of length n such that all edges of type I are on the odd places, while all the edges on the even places are of types Z or N. Let us introduce the same coordinate system as in the beginning of the solution of Problem 6, and also let us assume that the starting vertex of the path has the edge of type I going up, as in the figure in the solution of Problem 7. Then it follows that the y-coordinate of bee along the path will increase after every pair of edges by $\frac{\sqrt{3}}{2}$, while its x-coordinate will increase by $1/2$ after passing every pair IZ and decrease by $1/2$ after passing every pair IN. Note that there are no other possible pairs of edges. If n is odd, then at the end of the path

the bee will have to move by an edge of type I up. It follows that the y-coordinate of the bee at the end of the path depends only on n, and its x-coordinate will be equal to half of the difference between the number of edges of type Z and the number of edges of type N. In other words, it depends only on the number of edges of type Z (or N). It follows that for even n there are $\frac{n}{2}+1$ possibilities, and for odd n there are $\frac{n-1}{2}+1$ possibilities.

If the edges of type I are on the even places, then using the same coordinate system as for the previous case, we see that the y-coordinate of the bee will decrease by $\frac{\sqrt{3}}{2}$ after each pair of edges (and by 1/2 on the last step, if n is odd). The x-coordinate will increase by 1/2 after each pair NI and decrease by 1/2 after each pair ZI. It follows that the final vertex will depend only on the length n and on the difference between the numbers of the edges of types N and Z. Therefore, the number of possible final positions of the bee are $\frac{n}{2}+1$ for even n and $\frac{n+1}{2}+1$ for odd n. The sum of the two number of possibilities in the two cases is $n+2$ for all values of n (odd and even). The result will be the same, if we replace I by any of the types N or Z.

There are six possible types of shortest paths, depending on the parity of the places where a given type of edges occurs, and on the type of these edges. But there are paths belonging to two types simultaneously: $NININI\ldots$, $IZIZIZ\ldots$, and so on. It is easy to see that these paths are shortest, and each of them belongs to exactly two types. There are 6 such paths for every n bigger than 1: the ones starting with NI, IN, IZ, ZI, NZ, and ZN, respectively. If $n=1$, then we have three paths I, N, Z.

It follows that after a shortest path of length n the bee can reach $3(n+2)-6=3n$ vertices for all n bigger than 0. (Case $n=1$ has to be analyzed separately, but the answer is the same.)

Consequently, the total number of vertices that can be reached by the bee after n or less steps is

$$1+3+6+9+12+\cdots+3n=\frac{3n^2+3n+2}{2}.$$

Note that the formula is also correct for $n=0$.