Solutions 2016 AB Exam

Texas A&M High School Math Contest
October 22, 2016

1. If (x, y) is a point on the circle $x^2 + y^2 = 1$ and the distance from (x, y) to $(0, 1)$ is $\frac{6}{5}$, what is the value of y?

   ANSWER: $\frac{7}{25}$

   Solution: $\sqrt{(x-0)^2 + (y-1)^2} = \frac{6}{5}$
   $x^2 + (y-1)^2 = \frac{36}{25}$
   $x^2 + y^2 - 2y + 1 = \frac{36}{25}$
   $1 - 2y + 1 = \frac{36}{25}$
   $2y = 2 - \frac{36}{25} = \frac{14}{25}$
   $y = \frac{7}{25}$

2. A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the y-intercepts of these lines?

   ANSWER: 160

   Solution: Let the two lines be $y = 2x+b$ and $y = 6x+c$. We have $30 = 80+b$ and $30 = 240+c$. Hence $80 + b = 240 + c$ and $|b - c| = b - c = 240 - 80 = 160$.

3. What is the remainder when the polynomial $p(x) = (x-2)^{2015}$ is divided by the polynomial $q(x) = x - 1$?

   ANSWER: -1

   Solution: By the division algorithm

   $$(x-2)^{2015} = q(x)(x-1) + r$$

   where r is a constant. Substituting 1 gives $r = (-1)^{2015} = -1$.

4. For each positive integer n let $s(n)$ be the sum of the digits of n. For example $s(23) = 5$, $s(1014) = 6$, etc. What is the minimum value of $s(44 \cdot n)$ where $n = 1, 2, 3, ...$?

ANSWER: 2

Solution: It is reasonable to start with $n = 5$. $s(44 \cdot 5) = s(220) = 4$. Multiply by 5 again and get $s(220 \cdot 5) = s(1100) = 2$. Since 44 does not divide a power of 10, this is as small as possible.

5. Find the number of distinct real numbers x which have the property that the median of the five different numbers $x, 6, 4, 1, 9$ is equal to their mean.

ANSWER: 3

Solution: The mean of $x, 6, 4, 1, 9$ is

$$\frac{x+1+4+6+9}{5} = \frac{x+20}{5} = \frac{x}{5} + 4.$$

The median of $x, 6, 4, 1, 9$ depends on the size of x. if $x < 4$ then $x, 1, 4, 6, 9$ has median 4, so we want $\frac{x}{5} + 4 = 4$ and $\underline{x = 0}$. If $x > 6$ the median is 6 and we want $\frac{x}{5} + 4 = 6$ or $\underline{x = 10}$. If x is between 4 and 6 the median is x and we want $\frac{x}{5} + 4 = x$, i.e. $\frac{4}{5}x = 4$, $\underline{x = 5}$. There are no other possible values of x.

6. Which integer is nearest in value to the sum

$$\frac{2007}{2999} + \frac{8001}{5998} + \frac{2001}{3999}?$$

ANSWER: 3

Solution: $\frac{2007}{2999} > \frac{2}{3}$ since $6021 > (2)(2999)$

$\frac{8001}{5998} > \frac{8}{6} = \frac{4}{3}$ since $24003 > 4(5998)$

$\frac{2001}{3999} > \frac{2}{4} = \frac{1}{2}$ since $4002 > 3999$.

So the sum is slightly larger than $\frac{2}{3} + \frac{4}{3} + \frac{1}{2} = 2\frac{1}{2}$ but is not as large as 3.

7. Find all numbers m such that the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose coordinates are both positive.

ANSWER: $-\frac{3}{2} < m < 1$ or $m \in (-\frac{3}{2}, 1)$

Solution: $y = x - 2$
$y = mx + 3$
gives $(m-1)x + 5 = 0$ and $x = \frac{-5}{m-1}$.
The y value of the intersection point is

$$y = -\frac{5}{m-1} - 2 = \frac{-5 - 2m + 2}{m-1} = \frac{-2m-3}{m-1}.$$

We want $x > 0$ and $y > 0$ so $\frac{-5}{m-1} > 0$ and $\frac{-2m-3}{m-1} > 0$.
This implies $m - 1 < 0$ and $-2m - 3 < 0$
$m < 1$ and $m > -\frac{3}{2}$.

8. A point (x, y) is called an integer point if both x and y are integers. How many points in the graph of

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$$

are integer points? For example, $(2, -4)$ is one such point.

ANSWER: 10

Solution: $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$
$\frac{1}{y} = \frac{1}{4} - \frac{1}{x} = \frac{x-4}{4x}$
$y = \frac{4x}{x-4} = \frac{4(x-4)+16}{x-4} = 4 + \frac{16}{x-4}$.
Since y needs to be an integer, then $\frac{16}{x-4}$ needs to be an integer, i.e. $x - 4$ divides 16 for integer values of x. There are 10 possibilities: $x - 4 = -16, -8, -4, -2, -1, 1, 2, 4, 8, 16$. Each gives an integer point.

9. Consider the equation $x^2 + px + q = 0$. If the roots of this equation differ by 2, find q in terms of p.

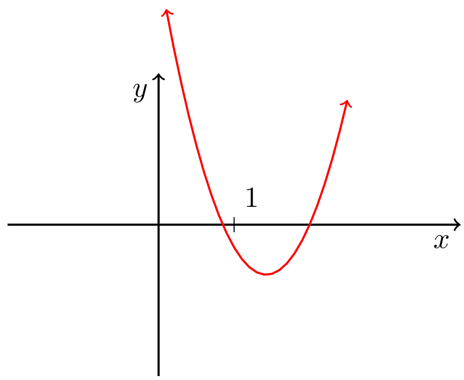
ANSWER: $q = \dfrac{p^2 - 4}{4}$

Solution: Let the roots be x_1 and $x - 2$. Then $x_, x_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$.
Thus, $x_1 - x_2 = \sqrt{p^2 - 4q} = 2$, or $p^2 - 4q = 4$. Hence

$$q = \frac{p^2 - 4}{4}.$$

10. The graph of $y = ax^2 + bx + c$ is as shown.How many of the following expressions are positive?

$$ab, \quad ac, \quad b, \quad a+b+c, \quad a-b+c$$



ANSWER: Two

Solution: The parabola opens up, so $a > 0$. The roots of y are positive, so $b < 0$ and $c > 0$. So $ab < 0$, $ac > 0$. The graph shows $y(1) < 0$ which means $a + b + c < 0$. It also shows $y(-1) > 0$ which means $a - b + c > 0$.

11. One and only one of the following integers does not divide

$$2^{1650} - 1.$$

Which integer is it?

(a) 3 (b) 7 (c) 31 (d) 127 (e) 2047

ANSWER: 127 or d

Solution: Note that each of the above numbers is one less than a power of two. That is,

$$3 = 2^2 - 1, \quad 7 = 2^3 - 1, \quad 31 = 2^5 - 1, \quad 127 = 2^7 - 1, \text{ and } 2047 = 2^{11} - 1$$

Moreover each of these exponents, with the exception of 7, divides 1650. Thus, the numbers 3, 7, 31, and 2047, divide $2^{1650} - 1$, while 127 does not. The following lines demonstrate the reasoning:

$$\begin{aligned} 2^{1650} - 1 &= \left(2^{11}\right)^{150} - 1 = (2047 + 1)^{150} - 1 = (k \cdot 2047 + 1) - 1 = k \cdot 2047, \text{ for some } k. \\ 2^{1650} - 1 &= \left(2^{7}\right)^{235} \cdot 2^5 - 1 = (127 + 1)^{235} \cdot 32 - 1 = (k \cdot 127 + 1) \cdot 32 - 1 = k \cdot 127 + 31, \text{ for some } k. \end{aligned}$$

12. Find the value of the product

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^3})(1 - \frac{1}{4^2})...(1 - \frac{1}{99^2})(1 - \frac{1}{100^2}).$$

ANSWER: $\frac{101}{200}$

Solution: The product can be written

$$\frac{2^2-1}{2^2}\frac{3^2-1}{3^2}\frac{4^2-1}{4^2}\cdots\frac{99^2-1}{99^2}\frac{100^2-1}{100^2}=$$
$$\frac{1\cdot 3}{2\cdot 2}\frac{2\cdot 4}{3\cdot 3}\frac{3\cdot 5}{4\cdot 4}\cdots\frac{98\cdot 100}{99\cdot 99}\frac{99\cdot 101}{100\cdot 100}=$$
$$\frac{(1\cdot 2\cdot 3\cdots 98\cdot 99)}{(1\cdot 2\cdot 3\cdots 99\cdot 100)}\frac{(3\cdot 4\cdot 5\cdots 101)}{(1\cdot 2\cdot 3\cdots 99\cdot 100)}=$$
$$\frac{1}{100}\cdot\frac{101}{2}=\frac{101}{200}.$$

13. The curve $y=x^4-8x^3+9x^2+20x+2$ intersects the line $y=2x+1$ at four distinct points. Find the average y-value of the intersection points.

ANSWER: 5

Solution: Let x_1,x_2,x_3,x_4 be the x-coordinates of the four points. Then x_1,x_2,x_3,x_4 are the roots of
$(x^4-8x^3+9x^2+20x+2)-(2x+1)=x^4-8x^3+9x^2+18x+1$.
So $x_1+x_2+x_3+x_4=8$. If $y_1,...,y_4$ are the corresponding y-coordinates, then $y_i=2x_i+1$ and

$$\begin{aligned} y_1+y_2+y_3+y_4 &= (2x_1+1)+(2x_2+1)+(2x_3+1)+(2x_4+1) \\ &= 2(8)+4=20. \end{aligned}$$

The average is $\dfrac{20}{4}=5$.

14. The sum of the squares of three prime numbers is 182. What is the sum of the three primes?

ANSWER: 18

Solution: Since 182 is even one of the primes must be 2. Let p and q be the odd primes. Then $2^2 + p^2 + q^2 = 182$ and $p^2 + q^2 = 178$. The only possible primes p are 3, 5, 7, 11, and 13 as $15^2 = 225 > 178$. The squares of these primes are shown below.

p	3	5	7	11	13
p^2	9	25	49	121	169

One of the primes must be 13 as $121 + 49 < 178$. Thus, the second prime must be 3, and the sum of the primes is $2 + 3 + 13 = 18$.

15. Duke had an average score of 85 on his first eight quizzes. He had an average score of 81 on his first nine quizzes. What score did he receive on his ninth quiz?

ANSWER: 49

Solution: Let $S_1, ..., S_8, S_9$ be the quiz scores. We are given $(S_1 + ... + S_8)/8 = 85$, so $S_1 + ... + S_8 = (8)(85) = 680$. Similarly $(S_1 + ... + S_9)/9 = 81$ and $S_1 + ... + S_8 + S_9 = (9)(81) = 729$. This means $680 + S_9 = 729$ and $S_9 = 49$.

16. We have a collection of balls, each labelled $1, 2, 3, ...,$ or 100. These balls are placed in a jar. After repeated trials it is determined that the probability of selecting a ball from the jar labelled n equals n times the probability of selecting a ball labelled 1. What is the probability of slecting a ball labelled 50?

ANSWER: $\frac{1}{101}$

Solution: Let $Pr(n)$ be the probability of selecting a ball labelled n. If $Pr(1) = x$, we are given

$$\begin{aligned} Pr(1) &= x \\ Pr(2) &= 2x \\ &\vdots \\ Pr(100) &= 100x. \end{aligned}$$

We must have

$$
\begin{aligned}
Pr(1) + Pr(2) + ... + Pr(100) &= 1 \\
x + 2x + ... + 100x &= 1 \\
(1 + 2 + ... + 100)x &= 1 \\
(50)(101)x &= 1 \\
x &= \frac{1}{50}\frac{1}{101}.
\end{aligned}
$$

So $Pr(50) = 50x = \dfrac{1}{101}$.

17. Find the smallest integer $n \geq 100$ such that $n^2 + 4n + 2$ is divisible by 7.

ANSWER: 100

Solution: Write $n = 7q + r$ with $r \in \{0, 1, ..., 6\}$.

$$
\begin{aligned}
n^2 + 4n + 2 &= (7q + r)^2 + 4(7q + r) + 2 \\
&= 7(7q^2 + 2qr + 4q) + r^2 + 4r + 2,
\end{aligned}
$$

and we want this to be divisible by 7. So we need 7 to divide r^2+4r+2. The only possibilities for r are $r = 1$ and $r = 2$, with q arbitrary. Since $n = 7q + r$ and $n \geq 100$ then q is at least 14. With $q = 14$ and $r = 2$ we have

$$n = 7(14) + 2 = 100.$$

18. How many 3-element subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are there such that they contain at least two consecutive integers?

ANSWER: 64

Solution: The number of subsets of three elements containing $0, 1$ is 8. Similarly for $1, 2$ etc. giving a total of $(9)(8) = 72$. But in this count, the 8 subsets $\{0, 1, 2\}, \{1, 2, 3\}, ...\{7, 8, 9\}$ have been counted twice. So $72 - 8 = 64$.

19. Find the smallest positive integer n such that when n is divided by 5 there is a remainder of 1 and when n is divided by 6 there is a remainder of 2.

ANSWER: 26

Solution: We have $n = 5s + 1 = 6t + 2$ for some integers s and t. Since $5s + 1 = 6t + 2$ then $5s - 6t = 1$. One solution to this equation is $s = -1$, $t = -1$, which gives $n = -4$. Since 30 is the least common multiple of 5 and 6 we obtain other possibilities for n, namely $n = -4 + 30k$. The least positive value for n is when $k = 1$, which gives $n = 26$.

20. Find the largest integer k such that 135^k divides 2016!
($n!$, n factorial, is defined as $n! = 1 \cdot 2 \cdots n$).

ANSWER: 334

Solution: Since $135 = 5 \cdot 27 = 5 \cdot 3^3$, then $135^k = 5^k \cdot 3^{3k}$. Let a be the number of $5's$ in the prime factorization of 2016! and b be the number of $3's$ in it. We need to find the largest integer k such that $k \leq a,\ 3k \leq b$. Consider all integers from 1 to 2016. Since $2016 = 5 \cdot 403 + 1$, there are 403 integers in that interval divisible by 5. Since $2016 = 25 \cdot 80 + 16$, there are 80 integers there divisible by $5^2 = 25$. Since $2016 = 125 \cdot 16 + 16$, there are 16 integers there divisible by $5^3 = 125$. Finally, since $2016 = 625 \cdot 3 + 141$ there are 3 integers there divisible by $5^4 = 625$. Overall the number of 5s in the prime factorization of 2016! is $403 + 80 + 16 + 3 = 502$. Analogously, let us compute b, the number of $3's$ in 2016!. Since

$$2016 = 3 \cdot 672 = 9 \cdot 224 = 27 \cdot 74 + 18 = 81 \cdot 24 + 72 = 243 \cdot 8 + 72 = 729 \cdot 2 + 558,$$

we conclude that $b = 672 + 224 + 74 + 24 + 8 + 2 = 1004$. Thus, $3k \leq 1004$, or $k \leq 334$. So $k \leq 334$ and also, $k \leq 502$, we conclude that $k = 334$ is the largest integer such that 135^k divides 2016!.