BC Exam Solutions
Texas A&M High School Math Contest
October 22, 2016

All answers must be simplified, and if units are involved, be sure to include them.

1. Given
$$A = \frac{1}{2-\sqrt{3}}$$
and
$$B = (\sqrt{5} - \sqrt{2}\sqrt{\sqrt{3}})(\sqrt{5} + \sqrt{2}\sqrt{\sqrt{3}}),$$
find $2A + B$ simplifying as much as possible.

Solution:
$$A = \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2+\sqrt{3}}{2^2-(\sqrt{3})^2} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$
and
$$B = (\sqrt{5})^2 - (\sqrt{2}\sqrt{\sqrt{3}})^2 = 5 - 2\sqrt{3}.$$
Therefore, $2A + B = 4 + 2\sqrt{3} + 5 - 2\sqrt{3} = 9$.

Answer: 9

2. Let x and y be the solutions of the system of equations
$$\sqrt{39 - 2x - 10y} = 5$$
and
$$\sqrt{15 - 2x + 2y} = 5.$$
Find $x + y$.

Solution: Squaring both sides of each equation, we get
$$\begin{cases} \sqrt{39-2x-10y} &= 5 \\ \sqrt{15-2x+2y} &= 5 \end{cases} \Leftrightarrow \begin{cases} 39-2x-10y &= 25 \\ 15-2x+2y &= 25 \end{cases} \Leftrightarrow \begin{cases} 2x+10y &= 14 \\ -2x+2y &= 10. \end{cases}$$
Adding the two equations we get $12y = 24$, which implies that $y = 2$. Replacing y in the first equation gives us $2x + 20 = 14$, which implies that $x = -3$. Therefore, $x + y = -1$.

Answer: -1

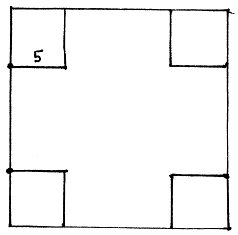
3. A collection of nickels and dimes has a total value of \$2.40 and contains 35 coins. How many nickels are in the collection?

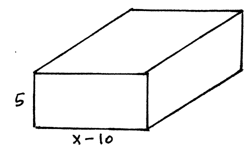
Solution: Let x be the number of nickels in the collection and y be the number of dimes in the collection. Then we have
$$\begin{cases} x+y &= 35 \\ 5x+10y &= 240 \end{cases} \Leftrightarrow \begin{cases} 5x+5y &= 175 \\ 5x+10y &= 240 \end{cases} \Leftrightarrow \begin{cases} x+y &= 35 \\ 5y &= 65 \end{cases} \Leftrightarrow \begin{cases} x &= 22 \\ y &= 13. \end{cases}$$

Answer: 22

4. A box containing 180 cubic inches is constructed by cutting from each corner of a cardboard square a small square with side 5 inches, and then turning up the sides. Find the area of the original piece of cardboard.

   Solution: Let x be the length (in inches) of the side of the cardboard square.





   The volume of the box is $5(x-10)^2$. We get that

$$5(x-10)^2 = 180 \Leftrightarrow (x-10)^2 = 36 \Rightarrow x - 10 = 6 \Leftrightarrow x = 16.$$

   The area of the original piece of cardboard is $16^2 = 256$ square inches.

   Answer: 256 in^2

5. Find the largest common divisor for the numbers

$$11^{100} + 11^{101} + 11^{102} + 11^{103}$$

   and

$$7^{100} + 7^{101} + 7^{102} + 7^{103}.$$

   Solution: We have that

$$\begin{aligned} 11^{100} + 11^{101} + 11^{102} + 11^{103} &= 11^{100}(1 + 11 + 11^2 + 11^3) = 11^{100}(1 + 11 + 121 + 1331) \\ &= 11^{100} \cdot 1464 = 11^{100} \cdot 8 \cdot 183 = 11^{100} \cdot 2^3 \cdot 3 \cdot 61 \end{aligned}$$

   and

$$\begin{aligned} 7^{100} + 7^{101} + 7^{102} + 7^{103} &= 7^{100}(1 + 7 + 7^2 + 7^3) = 7^{100}(1 + 7 + 49 + 343) \\ &= 7^{100} \cdot 400 = 7^{100} \cdot 2^4 \cdot 5^2. \end{aligned}$$

   Therefore, the largest common divisor for the two numbers is $2^3 = 8$.

   Answer: 8

6. Find the sum of all solutions of the equation

$$x^2 + 6x + \sqrt{x^2 + 6x} = 20.$$

   Solution: Let $y = \sqrt{x^2 + 6x}$. Then $y^2 + y = 20$ where $y \geq 0$. Solutions of $y^2 + y - 20 = 0$ are $y = -5$ or $y = 4$. Since $y \geq 0$, we conclude that $y = 4$. Thus,

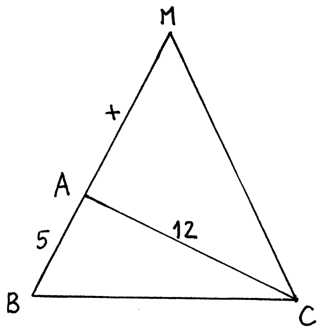
$$\sqrt{x^2 + 6x} = 4 \Leftrightarrow x^2 + 6x = 16 \Leftrightarrow x^2 + 6x - 16 = 0 \Leftrightarrow x = -8 \text{ or } x = 2.$$

   The sum of all real solutions of our initial equation is $-8 + 2 = -6$.

   Answer: -6

7. Suppose that $\triangle ABC$ is a right triangle with $\angle A = 90°$, $AB = 5$, and $AC = 12$. On the line AB we consider the point M such that $\triangle BMC$ is isosceles with $BM = CM$. Find AM.

Solution: Let $AM = x$.



Then $CM = BM = AB + AM = 5 + x$. By applying the Pythagorean Theorem in the right triangle CAM, we get that

$$CM^2 = AC^2 + AM^2 \Leftrightarrow (5+x)^2 = 12^2 + x^2 \Leftrightarrow x^2 + 10x + 25 = x^2 + 144 \Leftrightarrow 10x = 119 \Leftrightarrow x = 11.9.$$

Answer: 11.9

8. Mr. Kaye is 11 times as old as his daughter Lynn. Thirty-six years from now he will be at most twice as old as Lynn. At most, how old is Lynn?

Solution: Let x be the number of years in Lynn's age now. Mr. Kaye's age now is $11x$. Since Mr. Kaye's age in 36 years from now will be at most twice Lynn's age we get that

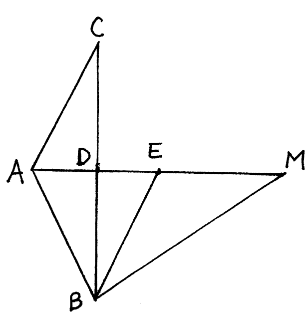
$$11x + 36 \leq 2(x+36) \Leftrightarrow 11x + 36 \leq 2x + 72 \Leftrightarrow 9x \leq 36 \Leftrightarrow x \leq 4.$$

Therefore, Lynn is at most 4 years old.

Answer: 4

9. In the $\triangle ABC$, $AB = AC$ and $\angle A = 120°$. The median AD to the side BC is extended through the point D with the segment $DM = 3AD$. Find $\angle DMB$.

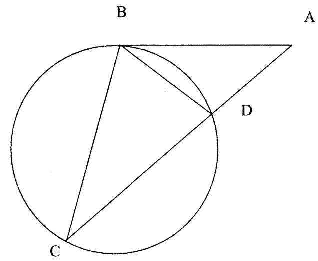
Solution:



First, since $\triangle ABC$ is isosceles, the median AD is also the altitude to BC and the bisector of $\angle A$. Let E be the midpoint of AM. Then $AE = EM = \dfrac{AM}{2}$, $AD = DE = \dfrac{AM}{4}$. In the $\triangle ABE$, BD then is the altitude to AE and the median, so $\triangle ABE$ is isosceles. Next, $\angle BAE = \angle BEA = 60°$, so $\triangle ABE$ is equilateral, thus $BE = AB = AE$. So $BE = EM$, and since $\angle E$ in the isosceles $\triangle BEM$ is equal to $120°$, $\angle EMB = \angle EBM = \dfrac{180° - 120°}{2} = 30°$. Finally, $\angle DMB = \angle EMB = 30°$.

Answer: 30°

10. In the figure below AB is tangent to the circle. If $AB = 8$ and AC exceeds AD by 12, what is AC?



Solution: We notice that $\angle ABD = \angle ACB$ and $\angle BAD = \angle CAB$. Thus, $\triangle ADB$ and $\triangle ABC$ are similar. This implies

$$\frac{AD}{AB} = \frac{AB}{AC} \Leftrightarrow AB^2 = AD \cdot AC.$$

Since $AB = 8$ and $AD = AC - 12$ we obtain that

$$8^2 = AC(AC - 12) \Leftrightarrow AC^2 - 12AC - 64 = 0 \Leftrightarrow AC = -4 \text{ or } AC = 16.$$

Since $AC > 0$, we conclude that $AC = 16$.

Answer: 16

11. Find the sum of all positive integers x for which $x + 56$ and $x + 113$ are perfect squares.

Solution: We have $x + 56 = p^2$ and $x + 113 = q^2$ with p and q positive integers. Clearly $p < q$. Subtracting the two equations we get

$$113 - 56 = q^2 - p^2 \Leftrightarrow (q-p)(q+p) = 57 \Leftrightarrow (q-p)(q+p) = 3 \cdot 19.$$

Since $0 < q - p < q + p$, we have the following two possible cases

$$\begin{cases} q - p &= 1 \\ q + p &= 57 \end{cases} \text{ or } \begin{cases} q - p &= 3 \\ q + p &= 19 \end{cases} \Leftrightarrow p = 28, q = 29 \text{ or } p = 8, q = 11 \Rightarrow x = 728 \text{ or } x = 8.$$

The sum is $728 + 8 = 736$.

Answer: 736

12. Consider the sum

$$S = \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}},$$

where n is a positive integer. If $S = 10$, what is the value of n?

Solution:

$$\begin{aligned} S &= \frac{\sqrt{2}-1}{(\sqrt{2}-1)(\sqrt{2}+1)} + \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} + \cdots + \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})} \\ &= \frac{\sqrt{2}-1}{(\sqrt{2})^2-1^2} + \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2} + \cdots + \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1})^2-(\sqrt{n})^2} \\ &= \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - 1. \end{aligned}$$

Therefore,

$$S = 10 \Leftrightarrow \sqrt{n+1} - 1 = 10 \Leftrightarrow \sqrt{n+1} = 11 \Leftrightarrow n + 1 = 121 \Leftrightarrow n = 120.$$

Answer: 120

13. Find the product of all solutions of the equation

$$(3x^2 - 4x + 1)^3 + (x^2 + 4x - 5)^3 = 64(x^2 - 1)^3.$$

Solution: If we denote $3x^2 - 4x + 1 = a$ and $x^2 + 4x - 5 = b$ then $a + b = 4x^2 - 4$ so our equation becomes

$$a^3 + b^3 = (a + b)^3 \Leftrightarrow a^3 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3 \Leftrightarrow 3a^2b + 3ab^2 = 0$$
$$\Leftrightarrow 3ab(a + b) = 0 \Leftrightarrow a = 0 \text{ or } b = 0 \text{ or } a + b = 0.$$

Then we have

$$a = 0 \Leftrightarrow 3x^2 - 4x + 1 = 0 \Leftrightarrow (3x - 1)(x - 1) = 0 \Leftrightarrow x = \frac{1}{3} \text{ or } x = 1$$
$$b = 0 \Leftrightarrow x^2 + 4x - 5 = 0 \Leftrightarrow (x - 1)(x + 5) = 0 \Leftrightarrow x = -5 \text{ or } x = 1$$
$$a + b = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = -1 \text{ or } x = 1.$$

The solutions of our equation are $-5, -1, \frac{1}{3}$, and 1. The product of all solutions is $\frac{5}{3}$.

Answer: $\frac{5}{3}$

14. A circle whose center is on the x-axis passes through the points $(3, 5)$ and $(6, 4)$. Find the radius of the circle.

Solution: Let $C(h, k)$ be the center of the circle and r be the radius of the circle. Since the center is on the x-axis we get that $k = 0$. Then an equation of the circle has the form $(x - h)^2 + y^2 = r^2$. The points $(3, 5)$ and $(6, 4)$ being on the circle gives us the system of equations

$$\begin{cases} (3 - h)^2 + 25 &= r^2 \\ (6 - h)^2 + 16 &= r^2. \end{cases}$$

If we subtract the two equations we get

$$(3 - h)^2 - (6 - h)^2 + 9 = 0 \Leftrightarrow 9 - 6h + h^2 - 36 + 12h - h^2 + 9 = 0 \Leftrightarrow 6h - 18 = 0 \Leftrightarrow h = 3.$$

So $(3 - 3)^2 + 25 = r^2 \Leftrightarrow r^2 = 25 \Leftrightarrow r = 5$.

Answer: 5

15. Find the sum of all integers N with the property that $N^2 - 71$ is divisible by $7N + 55$.

Solution: If $(N^2 - 71)/(7N + 55) = M$ where M is an integer, then $N^2 - 7MN - (55M + 71) = 0$. Solving for N, we find

$$N = \frac{7M \pm \sqrt{49M^2 + 220M + 284}}{2}.$$

The number under the radical must be a perfect square to produce an integer N. Notice that

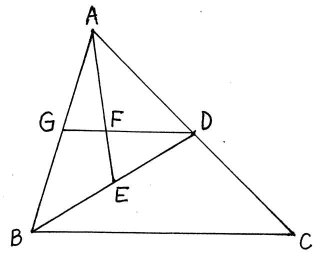
$$(7M + 15)^2 < 49M^2 + 220M + 284 < (7M + 17)^2$$

thus $49M^2 + 220M + 284 = (7M + 16)^2$. Then, solving $49M^2 + 220M + 284 = (7M + 16)^2$, we get $M = 7$, and so $N = 57$ or $N = -8$. The sum is $57 + (-8) = 49$.

Answer: 49

16. In the $\triangle ABC$, BD is the median to the side AC, DG is parallel to the base BC (G is the point of intersection of the parallel with AB). In the $\triangle ABD$, AE is the median to the side BD and F is the intersection point of DG and AE. Find $\dfrac{BC}{FG}$.

Solution:



Since DG is parallel to CB and $AD = DC$, we conclude that $BG = GA$. Thus DG is the midsegment of the $\triangle ABC$, which means that $GD = \frac{1}{2}BC$. In the $\triangle ABD$, DG and AE are two medians, and F is their intersection point, thus $FG = \frac{1}{3}DG$. Therefore, $FG = \frac{1}{6}BC$, which implies that $\dfrac{BC}{FG} = 6$.

Answer: 6

17. The function

$$f(x) = x^2 + (x+2)^2 + \cdots (x+98)^2 - \left[(x+1)^2 + (x+3)^2 + \cdots (x+99)^2\right]$$

is a linear function, $f(x) = ax + b$. Find $a - b$.

Solution: We notice that

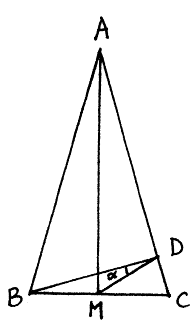
$$\begin{aligned} f(x) &= [x^2 - (x+1)^2] + [(x+2)^2 - (x+3)^2] + \cdots + [(x+98)^2 - (x+99)^2] \\ &= (x - x - 1)(x + x + 1) + (x + 2 - x - 3)(x + 2 + x + 3) + \cdots + (x + 98 - x - 99)(x + 98 + x + 99) \\ &= -(2x+1) - (2x+5) - \cdots - (2x+197) = -50 \cdot 2x - (1 + 5 + 9 + \cdots + 197) \\ &= -100x - 50 \cdot \frac{1+197}{2} = -100x - 4950. \end{aligned}$$

Therefore, $a = -100$, $b = -4950$, and $a - b = 4850$.

Answer: 4850

18. In the isosceles $\triangle ABC$ with $AB = AC$, let AM be the median to the side BC and let BD be the altitude to the side AC. If $\angle AMD = 4\angle BDM$ find $\angle ACB$.

Solution: Let $\angle BDM = \alpha$. Then $\angle AMD = 4\alpha$.



Since $\triangle BDC$ is a right triangle and M is the midpoint of the hypotenuse BC, we get that $DM = BM = MC$. This implies that $\triangle MDB$ is isosceles and thus $\angle MBD = \angle BDM = \alpha$.

Because $\triangle ABC$ is isosceles and M is the midpoint of BC we obtain that AM is perpendicular onto BC, which implies that $\angle BMD = 90° + 4\alpha$. In $\triangle MBD$ we have

$$\angle MBD + \angle BDM + \angle BMD = 180° \Leftrightarrow \alpha + \alpha + 90° + 4\alpha = 180° \Leftrightarrow \alpha = 15°.$$

In $\triangle AMD$ we have

$$\angle MAD = 180° - (\angle AMD + \angle ADM) = 180° - (4\alpha + 90° + \alpha) = 90° - 5\alpha = 15°.$$

Therefore, in $\triangle AMC$ we see that

$$\angle ACM = 90° - \angle MAC = 90° - \angle MAD = 90° - 15° = 75°,$$

which implies that $\angle ACB = 75°$.

Answer: 75°

19. Let f and g be two linear functions such that

$$f(x-1) = 2x - 3 + g(1) - f(1)$$

and

$$g(x-1) = 4x + 5 - g(1) - f(1),$$

for all numbers x. Find $g(5)$.

Solution: For $x = 2$ we get

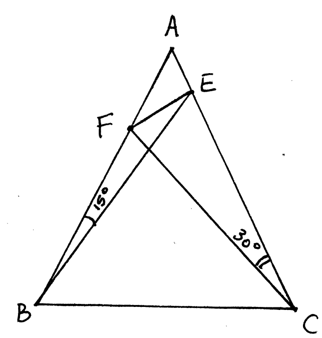
$$\begin{cases} f(1) = & 2\cdot 2 - 3 + g(1) - f(1) \\ g(1) = & 4\cdot 2 + 5 - g(1) - f(1) \end{cases} \Leftrightarrow \begin{cases} 2f(1) - g(1) & = 1 \\ f(1) + 2g(1) & = 13 \end{cases} \Leftrightarrow \begin{cases} f(1) & = 3 \\ g(1) & = 5. \end{cases}$$

So, $g(x-1) = 4x + 5 - 5 - 3 = 4x - 3 = 4(x-1) + 1$, which implies that $g(x) = 4x + 1$. Thus, $g(5) = 21$.

Answer: 21

20. Consider $\triangle ABC$ with $\angle B = \angle C = 70°$. On the sides AB and AC we take the points F and E, respectively, so that $\angle ABE = 15°$ and $\angle ACF = 30°$. Find $\angle AEF$.

Solution:



We see that

$$\angle EBC = \angle ABC - \angle ABE = 70° - 15° = 55°$$

and

$$\angle BCF = \angle ACB - \angle ACF = 70° - 30° = 40°.$$

In $\triangle BCE$ we have

$$\angle BEC = 180° - (\angle EBC + \angle ECB) = 180° - 125° = 55°.$$

So, $\angle BEC = \angle EBC = 55^\circ$, which implies that $\triangle BCE$ is isosceles. Thus $BC = EC$ (1).
In $\triangle BCF$ we have

$$\angle BFC = 180^\circ - (\angle FBC + \angle BCF) = 180^\circ - 110^\circ = 70^\circ.$$

Therefore, $\angle BFC = \angle FBC = 70^\circ$, which implies that $\triangle BCF$ is isosceles. We get that $BC = FC$ (2).
From (1) and (2) we deduce that $EC = FC$ which makes the $\triangle CEF$ isosceles. Since $\angle ECF = 30^\circ$, we get that

$$\angle FEC = \angle EFC = \frac{180^\circ - 30^\circ}{2} = 75^\circ,$$

and

$$\angle AEF = 180^\circ - \angle FEC = 180^\circ - 75^\circ = 105^\circ.$$

Answer: 105°