CD EXAM and Solutions
Texas A&M High School Math Contest
October 22, 2016

1. Let r be a real number such that

$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3.$$

Determine the value of

$$r^3 + \frac{1}{r^3}.$$

Solution. Let $u = \sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3$, $v = r + \frac{1}{r}$, and $w = r^3 + \frac{1}{r^3}$. Then $v = u^3 - 3u = 27 - 9 = 18$ and $w = v^3 - 3v = 5778$.

Answer. $\boxed{5778}$

2. The front tires of a car will wear out after 15,000 miles, while the rear tires will wear out after 25,000 miles. When should we switch the front and rear tires to maximize the distance we can drive with the car?

Solution. Let x be the distance at which the tires are rotated, front to back. After traveling a distance of y miles after the rotation, the relative wearing of the tires that were initially at front equals $x/15000+y/25000$, whereas the relative wearing of the tires that were initially at rear equals $x/25000 + y/15000$. The relative wearings cannot exceed one, so the maximum total distance $x + y$ occurs when the wearings are equal to each other, hence $x = y = 75000/8 = 9375$ miles. (This way, the car can travel 18750 miles.)

Answer. $\boxed{\text{9375 miles}}$

3. We have made a regular triangular pyramid with 120 congruent spheres. How many spheres fit in the basis of the pyramid?

Solution. The number a_n of spheres that can fill inside of an equilateral triangle with n balls on each side is $1+2+\cdots+n = n(n+1)/2$. We first need to find n such that $a_1+a_2+\cdots+a_n = 1+3+6+\cdots+n(n+1)/2 = 120$. In fact, $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 = 120$, so 36 spheres fit in the basis of the pyramid.

Answer. $\boxed{36}$

4. A non-constant real function f satisfies

$$f(x + y) = f(x) + f(y) - 2f(xy),$$

for all x, y. Find $f(2016)$.

Solution. It is clear that $f(0) = 0$ by plugging in $x = y = 0$. Also, $f(x + 1) = f(1) - f(x)$, so $f(x + 2) = f(1) - (f(1) - f(x)) = f(x)$, hence f is 2-periodic and $f(2016) = f(0) = 0$.

Answer. $\boxed{0}$

5. Find $f\left(\frac{1}{14}\right)+f\left(\frac{2}{14}\right)+f\left(\frac{3}{14}\right)+\cdots+f\left(\frac{13}{14}\right)$, where $f(x)=\frac{4^x}{4^x+2}$.

Solution. First, write $f(x)=(1+2^{1-2x})^{-1}$, and conclude $f(1-x)=(1+2^{2x-1})^{-1}$. Indeed, simple arithmetic shows that $f(x)+f(1-x)=1$, hence the sum equals $6(1)+f(7/14)=6.5$.

Answer. Either of the following equal numbers: $13/2, 6.5$

6. How many natural numbers n exist such that

$$1!+2!+3!+\cdots+n!$$

is a complete square?

Solution. Simple examination shows that for the $n=1$ and $n=3$ the sum is a complete square. However, $n>4$ gives a sum whose ones digit is always 3 because $1!+2!+3!+4!=33$ and $n!$ is a multiple of 10 for $n>4$. Therefore, the sum cannot be a complete square for any other values of n than 1 and 3.

Answer. 2

7. An accurate analog clock has been wrongly designed in that its minute and hour hands are indistinguishable from each other. In the 12-hour period from noon till (but before) midnight, how many moments are there when it is not possible to immediately tell the time on this clock?

Solution. Let $0 \le x, y < 12$ denote the coordinates of the hour and minute hands on the clock face, respectively. Note that time is completely determined by the coordinate x; the coordinate y is just to help with more accurate reading of the minutes by an enlarging factor of 12. In fact, because the minute hand moves 12 times faster than the hour hand, we will have $y=12\{x\}$, where $\{x\}=x-[x]$ is the fractional part of x. Since the clock and minute hands in the problem are indistinguishable, the clock would look ambiguous if $y \ne x$ and $x=12\{y\}$, so that time x is just as likely as time y. This means we have to find the number of solutions of the system

$$y=12\{x\}, \quad x=12\{y\}, \quad (y \ne x).$$

Eliminating y, the equations reduce to $x=12(12x-12[x])-12[12x-12[x]]=144x-12[12x]$, hence $\frac{143}{12}x=[12x]=k$, where k is a nonnegative integer at most 143. This leads to a parametric solution of the form $x=12k/143$, and a simple verification shows that these are, indeed, all solutions of the reduced equation provided $k \in \{0,1,\ldots,142\}$. Among these 143 moments, there are precisely 11 moments when the hands coincide ($y=x$), so there is a total of $143-11=132$ ambiguous moments. The moments when the two hands coincide may be found by setting $x=12\{x\}$, which gives $k=13[12k/143]$. In summary, the ambiguous moments $x=12k/143$ are parametrized by those values of k in $\{0,1,\ldots,142\}$ that are not multiples of 13 (i.e., $0,13,26,39,\ldots,130$).

Answer. 132

8. Find all real values of x such that

$$\left(\sqrt{4-\sqrt{15}}\right)^x+\left(\sqrt{4+\sqrt{15}}\right)^x=8.$$

Solution. Note that $(4-\sqrt{15})(4+\sqrt{15}) = 1$. Letting $a = \left(\sqrt{4-\sqrt{15}}\right)^x$ implies $a + 1/a = 8$, or equivalently, $a^2 - 8a + 1 = 0$. Therefore, $a = 4 \pm \sqrt{15} > 0$. It is clear that the only roots of the equation $(\sqrt{4-\sqrt{15}})^x = 4 \pm \sqrt{15}$ are $x = \mp 2$.

Answer. $\boxed{\pm 2}$

9. Find the sum of all positive numbers x such that $\sqrt[3]{2+x} + \sqrt[3]{2-x}$ is an integer.

Solution. Such integer $k = \sqrt[3]{2+x} + \sqrt[3]{2-x}$ cannot be zero, and satisfies

$$[(k^3-4)/k]^3 = 27(4-x^2) < 108.$$

The only integers with this property are $k = 1, 2$. The corresponding positive value for x is $\sqrt{4-[(k^3-4)/(3k)]^3}$.

Answer. Either of the following equal numbers:
$\boxed{\sqrt{5}+10/\sqrt{27}, (\sqrt{135}+10)/\sqrt{27}, \sqrt{5}+10/(3\sqrt{3}), (3\sqrt{15}+10)/(3\sqrt{3})}$

10. What is the 2016th digit after the decimal point of the number $(\sqrt{31}+\sqrt{32})^{2016}$?

Solution. Let $k = (\sqrt{31}+\sqrt{32})^{2016} + (\sqrt{31}-\sqrt{32})^{2016}$. Binomial expansions of both terms reveal the fact that the odd powers cancel, and the terms with even powers are all integers, hence k is a positive integer. On the other hand, $|\sqrt{31}-\sqrt{32}| = 1/(\sqrt{31}+\sqrt{32}) < 1/(2\sqrt{31}) < 1/10$, so $0 < (\sqrt{31}-\sqrt{32})^{2016} < 10^{-2016}$, so all the first 2016 digits after the decimal point of $(\sqrt{31}-\sqrt{32})^{2016}$ are 0's (but the fraction is nonzero), hence all the first 2016 digits after the decimal point of $(\sqrt{31}+\sqrt{32})^{2016}$ are 9's.

Answer. $\boxed{9}$

11. Two points A, B are 4 miles apart, and the midpoint between them is the center of a circular lake of diameter 2 miles. What is the shortest distance for an ant that wants to travel from A to B?

Solution. The shortest distance consists of two congruent line segments that are tangent to the circle and start from the two points, as well as an arc of the unit circle. The length of each line segment is $\sqrt{2^2-1^2} = \sqrt{3}$ and the arc is 60 degrees, so its length is $\pi/3$.

Answer. $\boxed{2\sqrt{3}+\pi/3 \text{ miles}}$

12. Find the smallest number k such that the inequality

$$\left|\frac{a+b}{2} - \sqrt{ab}\right| \leq k\,|a-b|$$

holds for all positive numbers a, b.

Solution. Define $x > 0$ such that $b = ax^2$. Then the inequality reduces to $|1+x^2-2x| \leq 2k|1-x^2|$. Without loss of generality, suppose $x \neq 1$. Dividing by $|x-1|$ the inequality reads $|x-1| \leq 2k|x+1|$. Geometrically, we are comparing the distances of x to the points ± 1 on the x-axis, and since x is larger than the midpoint 0 between -1 and 1, x will always be closer to 1 than -1, so $k = 1/2$ will always guarantee the inequality. On the other hand, if we choose any number $k < 1/2$, the inequality will not hold for small (or large) enough x, e.g., any $x \in (0, \frac{1-2k}{1+2k})$ would not satisfy the inequality.

Answer. $\boxed{1/2}$

13. Solve the equation $x^3 + x^2 + x = -\dfrac{1}{3}$.

Solution. A substitution $x = 1/y$ implies $y^3 + 3y^2 + 3y + 3 = (y+1)^3 + 2 = 0$, so $y = -1 - \sqrt[3]{2}$ and $x = -1/(1 + \sqrt[3]{2})$.

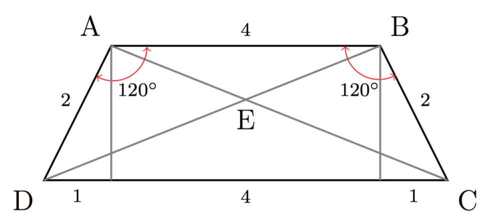
Answer. $\boxed{x = -1/(1 + \sqrt[3]{2}) = (-1 + \sqrt[3]{2} - \sqrt[3]{4})/3}$

14. A trapezoid $ABCD$ is isosceles, with $\overline{AD} = \overline{BC} = 2, \overline{AB} = 4$, and $\angle ABC = \angle BAD = 120°$. Let E be the intersection of the two diagonals. Find the ratio

$$\frac{\overline{BE}}{\overline{DE}}.$$

Solution. We have $\angle C = \angle D = 180° - 120° = 60°$, hence the vertical projections of the legs AD and BC onto the base CD are half of their lengths. This means $\overline{CD} = \overline{AB} + 1 + 1 = 6$. Now, the triangles ABE and CDE are similar, so

$$\frac{\overline{BE}}{\overline{DE}} = \frac{\overline{AB}}{\overline{CD}} = \frac{4}{6} = \frac{2}{3}.$$



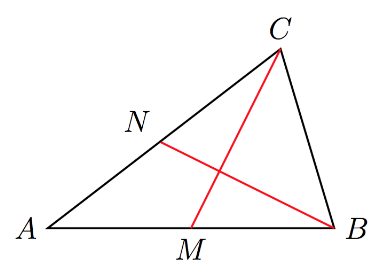
Answer. $\boxed{\text{Either of the following equal numbers: } 4/6, 2/3}$

15. In a triangle ABC the medians from vertices B and C are perpendicular. If $\overline{AB} = 19$ and $\overline{AC} = 22$, what is $\overline{BC}$?

Solution. Let those perpendicular medians be BN and CM, and suppose they intersect at O. By Pythagorean theorem we have

$$\overline{MN}^2 + \overline{BC}^2 = \overline{ON}^2 + \overline{OM}^2 + \overline{OB}^2 + \overline{OC}^2 = \overline{BM}^2 + \overline{CN}^2.$$

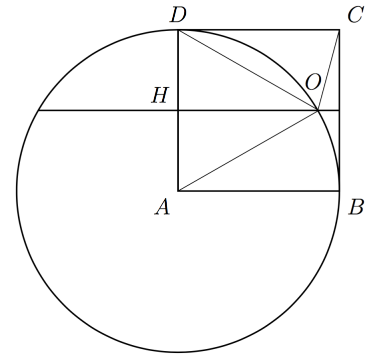
Since M, N are the midpoints of the sides AB, AC, respectively, we have $\overline{MN} = \overline{BC}/2, \overline{BM} = \overline{AB}/2 = 19/2$, and $\overline{CN} = \overline{AC}/2 = 11$. We conclude that $5\overline{BC}^2/4 = 11^2 + (19/2)^2$, so $\overline{BC}^2 = (22^2 + 19^2)/5 = 169$, hence $\overline{BC} = 13$.



Answer. $\boxed{13}$

16. A square $ABCD$ is given. A circle with radius AB and center A is drawn. This circle intersects the perpendicular bisector of BC in two points, of which O is the closest to C. Find the value of $\angle AOC$ in degrees.

Solution. Let the bisector of BC intersect AD at H. In the right triangle AOH we know $\overline{AH}/\overline{AO} = \overline{AH}/\overline{AB} = 1/2$, hence $\angle AOH = 30°$. Therefore, $\angle HOD = 30°$, by symmetry. Also, $\angle OAB = 30°$ because $AB \parallel OH$ and AO is a transversal. This means in the isosceles triangle OAB, we have $\angle BOA = \angle OBA = (180 - 30)/2 = 75°$, so by symmetry $\angle DOC = 75°$. Consequently, we have

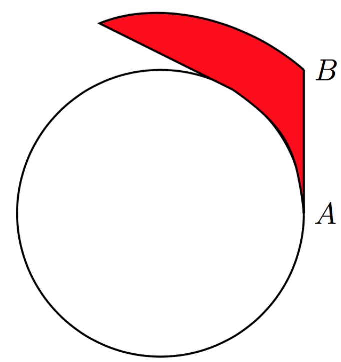


$$\angle AOC = \angle AOH + \angle HOD + \angle DOC = 30° + 30° + 75° = 135°.$$

Answer. $\boxed{\text{135 degrees}}$

17. A line segment AB of length 10 is tangent to a circle of radius 10 at the point A. If we rotate the circle and the tangent line 60 degrees about the center of the circle, what is the area of the region swept out by AB?

Solution. Let O be the center of the circle. By Pythagorean theorem, we have $\overline{OB} = \overline{OA}^2 + \overline{AB}^2 = 10^2 + 10^2 = 200$, so B moves along a circle during the rotation. In a full rotation scenario the region swept by AB is an annulus with area $\pi(200) - \pi(100) = 100\pi$. Since the swept area is proportional to the angle of rotation, the area in questions is $60/360$ of the area of the annulus, i.e., $100\pi/6$.



Answer. $\boxed{\text{Either of the following equal numbers: } 100\pi/6, 50\pi/3}$

18. A frog and a grasshopper are 2 meters apart. Each second, the frog moves either 25 cm or 50 cm on the ground towards the grasshopper, and also the grasshopper jumps either 25 cm or 50 cm towards the frog. The frog will eat the grasshopper if the two reach each other on the ground. In how many ways can the grasshopper be eaten by the frog?

Solution. Take 25 cm as the unit of length, so the distance between the frog and grasshopper is 8. Let f_k, g_k denote the distance - either 1 or 2 - traveled by the frog and grasshopper from $t = k - 1$ to $t = k$ seconds. We want to find the number of solutions of the equation

$$\sum_{k=1}^{N} f_k + g_k = 8.$$

Because $1 \le f_k, g_k \le 2$, the only possibilities for N are 2, 3, 4. The cases $N = 2, 4$ are two extreme cases; $N = 2$ happens only if $f_1 = f_2 = g_1 = g_2 = 2$, and $N = 4$ happens only if $f_1 = \cdots = f_4 = g_1 = \cdots = g_4 = 1$. To find the number of solutions for $N = 3$ we set $f'_k = f_k - 1$, $g'_k = g_k - 1$, so $f'_k, g'_k \in \{0, 1\}$, and the equation reduces to

$$f'_1 + f'_2 + f'_3 + g'_1 + g'_2 + g'_3 = 2.$$

This is precisely the number 6-digit binary codes made by two 1's and four 0's, which is the binomial coefficient $\frac{6!}{4!2!} = 15$. So the total number of cases in problem is $15 + 1 + 1 = 17$ by the rule of sum.

Answer. $\boxed{17}$

19. Solve the equation $x = \sqrt{a - \sqrt{a + x}}$, where $a > 0$ is a parameter.

Solution. Note that we must have $x \geq 0$ and $\sqrt{a + x} \leq a$. Squaring both sides of the given equation we get $x^2 = a - \sqrt{a + x}$, hence $x^2 \leq a$ and $(a - x^2)^2 = a + x$, or equivalently,

$$a^2 - a(2x^2 + 1) + x^4 - x = 0.$$

Now, solve this quadratic equation for the parameter a. The discriminant is $(2x^2+1)^2 - 4(x^4 - x) = (2x+1)^2$, and since $x \geq 0$, we find $a = [(2x^2 + 1 \pm (2x + 1)]/2$. The minus sign gives $a = x^2 - x$, which contradicts $x^2 \leq a$. The plus sign leads to $a = x^2 + x + 1$, giving $x = (-1 \pm \sqrt{4a - 3})/2$, so the only nonnegative choice is $x = (-1 + \sqrt{4a - 3})/2$, which occurs if and only if $4a - 3 \geq 1$, if and only if $a \geq 1$.

Answer. $\boxed{x = (-1 + \sqrt{4a - 3})/2}$