DE EXAM
Texas A&M High School Math Contest
November 2016

1. Find the length of an edge of an equilateral triangle that has one corner at $(0, 0)$ and the other two on the graph of $xy = 1$. Simplify fully.

2. In quotient and remainder division of one positive integer p by another, d, the quotient q is the largest integer q such that $qd \leq p$, and the remainder is $p - qd$. Thus, for instance, the remainder when dividing 22 by 5 is 2, and the quotient is 4.

   Find the smallest positive integer n so that when n is divided by 3, the remainder is 1, when n is divided by 5, the remainder is 2, when n is divided by 7, the remainder is 3, and when n is divided by 11, the remainder is 4.

3. Multiplying $(1 + x + x^2 + \cdots + x^9)^{10}$ gives an expression

   $$c_0 + c_1x + c_2x^2 + \cdots + c_{90}x^{90} = 1 + 10x + 55x^2 + \cdots + 10x^{89} + x^{90}.$$

   Find $c_0 + c_1 + c_2 + \cdots + c_{90}$.

4. Let m be the integer whose binary (base 2) representation is 1010101. Find the binary representation of m^2.

5. Find the (unique) y so that $1/2 < y$ and

   $$\frac{y^2}{\sqrt{1 - y^2}} + \sqrt{1 - y^2} = 2y.$$

6. Solve for x and y:

   $$\begin{aligned} \log_2(x^3y^4) &= -2 \\ \log_4(x^5y^7) &= -1. \end{aligned}$$

7. Let P be a polynomial of degree 5. Given that $P(0) = 0$, and $P(1) = P(2) = P(3) = P(4) = P(5) = 1$, find $P(8)$.

8. Let $\theta = \arctan 2 + \arctan 3$. Find $1/\sin^2\theta$ and simplify fully.

9. Let $f(x) = |x| + |2x - 1| - |3x - 2|$. What is the area of the region inside the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ and above the graph of $f(x)$? Write the answer as an improper fraction in lowest terms.

10. How many integer pairs (m, n) are there so that $0 \leq n \leq \sqrt{2}m$ and $m \leq 10$?

11. Find the sum of $5!/(a!b!c!)$ over all lists (a, b, c) of nonnegative integers so that $a + b + c = 5$.

12. Find a and b, with $b > 0$, so that $(x - b)^2$ is a factor of $x^4 - ax + 1$.

13. Find the exact value of $\tan \pi/8$ and simplify.

14. How many real numbers x are there such that $\sin 2x + \cos 2x = 1 + \cos 3x$ and $0 < x < 2\pi$?

15. Let C be the unit cube with corners such that each coordinate is 0 or 1. (Thus, $(0, 0, 0)$ and $(1, 1, 1)$ are a pair of opposite corners.) Let H be the set of all points inside C and equally distant from those two corners. Let T be that part of the cube consisting of all points that are on some line segment joining $(1, 1, 1)$ to a point in H. Find the volume of T.

16. Find integers A, B, C, D so that

$$\cos 3x = A \cos^3 x + B \cos^2 x + C \cos x + D.$$

17. Given that $x + y = A$ and $x^2 + y^2 = B$, express $x^4 + y^4$ in the form $PA^4 + QA^2B + RB^2$. That is, find values for P, Q, and R that ensure the identity holds whatever the values of x and y.

18. Find the least prime number that divides $10! + 1$.

19. Two vertices of a triangle are $(0, 0)$ and $(1, 0)$. The third is taken at random from the line segment from $(-1, 1)$ to $(2, 1)$. What is the probability that the triangle entirely encloses the circle about $(1/2, 1/2)$ of radius $1/10$?