2016 EF Exam

Texas A&M High School Students Contest

Solutions

October 22, 2016

1. Assume that p and q are real numbers such that the polynomial $x^4 + 1$ is divisible by $x^2 + px + q$. Find $\left|\frac{q}{p}\right|$.

2. What is the maximal value of the expression $\frac{1}{a + \frac{1000}{b+\frac{1}{c}}}$, where a, b, and c are pairwise distinct positive (decimal) digits (write you answer in terms of $\frac{p}{q}$).

3. A cube of size 20 is subdivided into 8000 unit boxes and a number is written inside of each of these unit cubes. It is known that the sum of all numbers in each array of 20 boxes which is parallel to an edge of the cube is equal to 1. The number 10 is written in one of the unit cubes. Consider the three layers of size $1 \times 20 \times 20$, parallel to the faces of the cube and containing the unit cube with 10 in it. Find the sum of all numbers that are in the boxes outside of these three layers.

4. How many positive integers n exist such that
$$1! + 2! + 3! + \cdots + n!$$
is a complete square?

5. Assume that a pentagon $AKHIL$ can be inscribed into a circle. Let U be the intersection of diagonals AH and KI, and M be the intersection of UL and AI. Furthermore, assume that $|HU| = 4|KU|$, $|UM| = 2|LM|$, and $[AUK]+[HUI] = 5$, where $|\cdot|$ denote the length of the corresponding segment and the square brackets denote the area of the corresponding polygon. Compute $\sqrt{[HUK] \cdot [ILAU]}$.

6. Let $f(x) = \frac{1}{x}$. Given a positive real number a find the distance between the parallel tangents to the graph of f at $x = a$ and $x = -a$.

7. Find the minimal positive $x + y$ such that $(1 + \tan x)(1 + \tan y) = 2$.

8. In how many regions do n straight lines divide the plane if any two lines are not parallel and no three lines have a common point?

9. Real numbers a, c are independently chosen uniformly at random from the open interval $(0, 1)$. What is the probability that the quadratic equation $ax^2 + x + c = 0$ has real roots?

10. Evaluate

$$\sqrt[3]{1\sqrt[3]{2\sqrt[3]{4\sqrt[3]{8\sqrt[3]{16\sqrt[3]{\cdots}}}}}}$$

11. John placed 2017 letters A,B, and C in a row in such a way that at least one other letter lies between any two A's, at least two other letters lie between two B's , and at least three other letters lie between any two C's. What is the maximal possible number of C's in this row of letters?

12. Among all $a > 1$ find the value of a for which the equation $a^x = \log_a x$ has exactly one solution.

13. A sequence $x_1, x_2, x_3, \ldots$ is called periodic if there exists a positive integer T such that $x_{n+T} = x_n$ for any n and the minimal T satisfying this property is called the period of this sequence. Suppose that two sequences with period 7 and 13 coincide up to their kth elements (including the kth elements). What is the maximal possible k?

14. Regular hexagon ψ has side length 1. What is the area of the union of all regular hexagons Ψ in the same plane as ψ such that each vertex of ψ lies on a different side of Ψ?

15. Evaluate the following integral

$$\int_0^{\pi} \Big(\sin x + \sin(2x) + \sin(3x) + \ldots + \sin(2016x)\Big)^2\, dx.$$

16. Assume that a function $f(x)$ is twice differentiable on the interval $[0, 1]$, $f(0) = f(1) = 0$ and $|f''(x)| \leq 1$ on $[0, 1]$. Let $M(f)$ be the maximal value of f on the interval $[0, 1]$. What is the maximal possible value of $M(f)$ among all functions f satisfying the properties above?

17. Find the positive integer k for which $A_k = \dfrac{10^k + 99^k}{k!}$ has the maximal value.

18. What is the maximal number of factors of the form $\sin \frac{\pi n}{x}$ that can be erased on the left-hand side of the equation

$$\sin\frac{\pi}{x}\sin\frac{2\pi}{x}\ldots\sin\frac{2016\pi}{x} = 0$$

so that the number of positive integer solutions will not change.

19. What is the 2016th digit after the decimal point of the number $(\sqrt{31} + \sqrt{32})^{2016}$?