2016 EF Exam
Texas A&M High School Students Contest
Solutions
October 22, 2016

1. Assume that p and q are real numbers such that the polynomial x^4+1 is divisible by x^2+px+q. Find $\left|\frac{q}{p}\right|$.

   Answer $\boxed{\frac{1}{\sqrt{2}}}$

   Solution 1 (without knowledge of complex numbers) It is easy to factor x^4+1 into two quadratic factors:

   $$x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$

   For both factors of this factorization $\left|\frac{q}{p}\right| = \frac{1}{\sqrt{2}}$. However, we also have to justify that such factorization is unique up to a permutation of the factors. For this one can use long division of the polynomials x^4+1 and x^2+px+q to obtain that x^4+1 is divisible by x^2+px+q if and only if $p^2=2q$ and $q(p^2-q)=1$. If p is real, then from the first relation $q\geq 0$ and substituting the first relation into the second one we get $q^2=1$, so $q=1$. Therefore $p^2=2$, i.e. $p=\pm\sqrt{2}$, q.e.d.

   Solution 2 (with knowledge of complex numbers) Factor x^4+1 into linear factors with complex coefficients by solving $x^4+1=0 \Leftrightarrow x^4=-1$, i.e. we have to find all roots of order 4 of -1. They are

   $$x_1 = \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, \quad x_2 = -\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, \quad x_3 = -\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}, \quad x_4 = \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}.$$

   So, $x^4+1=(x-x_1)(x-x_2)(x-x_3)(x-x_4)$ and x^2+px+q divides x^4+1 if and only if it is of the form $(x-x_k)(x-x_l)$, $k\neq l$. Moreover, the coefficients p and q are real if and only if x_k and x_k are complex conjugate. Therefore the required x^2+px+q is either equal to $(x-x_1)(x-x_4)=x_2-\sqrt{2}x+1$ or $(x-x_2)(x-x_3)=x^2+\sqrt{2}x+1$ which implies that $\left|\frac{q}{p}\right|=\frac{1}{\sqrt{2}}$.

2. What is the maximal value of the expression $\frac{1}{a+\frac{1000}{b+\frac{1}{c}}}$, where a, b, and c are pairwise distinct positive (decimal) digits (write you answer in terms of $\frac{p}{q}$).

   Answer $\boxed{\frac{1}{102}}$.

   Solution Since a,b,c are positive integers not exceeding 9, for any a and c we have $\frac{1}{a+\frac{1000}{b+\frac{1}{c}}} \leq \frac{1}{a+\frac{1000}{9+\frac{1}{c}}}$. Since a and c must be distinct the maximum of the expression is realized in on of the following two cases

(a) $a = 1$ and $c = 2$:in this case $\dfrac{1}{1+\frac{1000}{9+\frac{1}{2}}} = \dfrac{19}{2019}$

(b) $a = 2$ and $c = 1$:in this case $\dfrac{1}{2+\frac{1000}{9+1}} = \dfrac{1}{102}$

Note that $\dfrac{1}{102} > \dfrac{19}{2019}$, because $102 \cdot 19 = 1938 < 2019$.

3. A cube of size 20 is subdivided into 8000 unit boxes and a number is written inside of each of these unit cubes. It is known that the sum of all numbers in each array of 20 boxes which is parallel to an edge of the cube is equal to 1. The number 10 is written in one of the unit cubes. Consider the three layers of size $1 \times 20 \times 20$, parallel to the faces of the cube and containing the unit cube with 10 in it. Find the sum of all numbers that are in the boxes outside of these three layers.

Answer $\boxed{333}$

Solution Denote the chosen unit cube (where 10 is written) by K. It belongs two one horizontal layer H and two vertical layers. The sum of all numbers in the 361 vertical columns, which do not lie in these vertical layers, is equal to 361. From this sum we have to subtract the sum S of numbers, lying in the intersections of these vertical columns with the layer H. The latter intersection lies in 19 parallel array belonging to H and not containing K. The sum of all numbers in these 19 arrays is equal to 19 and it exceeds S by the sum of 19 numbers, lying in the array in H, containing K and perpendicular to those 19 array previously chosen in H (i.e. the sum of all numbers in this array except the number in the unit cube K). The latter sum is obviously equal to $1 - 10 = -9$. So, $S = 19 - (-9) = 28$ and finally the desired sum is $361 - 28 = \boxed{333}$.

4. How many positive integers n exist such that

$$1! + 2! + 3! + \cdots + n!$$

is a complete square?

Answer $\boxed{2}$

Solution Note that for $n \geq 5$ the last digit of $n!$ is 0, therefore for $n \geq 5$ the last digit of the sum above is the same as the last digit of $1! + 2! + 3! + 4! = 33$, i.e. it is equal to 3. No complete square has 3 as the last digit, so for $n \geq 5$ the sum above cannot be a complete square. Among $n = 1, 2, 3, 4$ only for $n = 1$ and $n = 3$ the sum above is a complete square, so we have only 2 positive integers like this.

5. Assume that a pentagon $AKHIL$ can be inscribed into a circle. Let U be the intersection of diagonals AH and KI, and M be the intersection of UL and AI. Furthermore, assume that $|HU| = 4|KU|$, $|UM| = 2|LM|$, and $[AUK] + [HUI] = 5$, where $|\cdot|$ denote the length of the corresponding segment and the square brackets denote the area of the corresponding polygon. Compute $\sqrt{[HUK] \cdot [ILAU]}$.

Answer $\boxed{\dfrac{10\sqrt{6}}{17}}$

Solution From the fact that the angles inscribed in a circle and subtended by the same chord are equal it follows that the triangles $\triangle HUI$ and $\triangle KUA$ are similar and from the fact that $|HU| = 4|KU|$ it follows that their sides are in a ratio of 4 to 1. Then $[HUI] = 16 \cdot [AUK]$, so

$[HUI] + [AUK] = 5 \implies 16 \cdot [AUK] + [AUK] = 5 \implies [AUK] = \frac{5}{17}$, and $[HUI] = \frac{80}{17}$. Now, we note that $\frac{[AUK]}{[HUK]} = \frac{[AUI]}{[HUI]} = \frac{AU}{HU}$, so $[HUK] \cdot [AUI] = [AUK] \cdot [HUI] = \frac{5}{17} \cdot \frac{80}{17} = \frac{400}{289}$. Finally, $[ILAU] = [AUI] + [ALI]$, and $\frac{[AUI]}{[ALI]} = \frac{UM}{LM} = 2$, so $[ILAU] = [AUI] + \frac{1}{2} \cdot [AUI] = \frac{3}{2} \cdot [AUI]$, and $[HUK] \cdot [ILAU] = \frac{3}{2} \cdot [HUK] \cdot [AUI] = \frac{600}{289}$. Then $\sqrt{[HUK] \cdot [ILAU]} = \boxed{\dfrac{10\sqrt{6}}{17}}$.

6. Let $f(x) = \frac{1}{x}$. Given a positive real number a find the distance between the parallel tangents to the graph of f at $x = a$ and $x = -a$.

   Answer: $\boxed{\dfrac{4a}{\sqrt{a^4+1}}}$

   Solution At each of these points, the slope of the tangent lines is $-\frac{1}{a^2}$. Since the graph of f is symmetric about the origin, we can simply find the distance from the origin to one of tangent lines and double it. The tangent line to the graph at a has equation $y = \frac{1}{a} - \frac{1}{a^2}(x-a) = \frac{2a-x}{a^2}$. Equivalently it can be written as $x + a^2y - 2a = 0$. Note that the distance from a point with coordinates (x_0, y_0) to a line given by the equation $\alpha x + \beta y + \delta = 0$ is equal to $\dfrac{|\alpha x_0 + \beta y_0 + \delta|}{\sqrt{\alpha^2+\beta^2}}$. Applying this formula to $(x_0, y_0) = (0, 0)$ and our tangent line $x + a^2y - 2a = 0$ we get that the distance from the origin to this line is equal to $\dfrac{2a}{\sqrt{a^4+1}}$, which implies our answer after doubling.

7. Find the minimal positive $x + y$ such that $(1 + \tan x)(1 + \tan y) = 2$.

   Answer $\boxed{\dfrac{\pi}{4}}$

   Solution $(1 + \tan x)(1 + \tan y) = 1 + \tan x + \tan y + \tan x \tan y = 2 \Rightarrow$

   $$\tan x + \tan y = 1 - \tan x \tan y. \tag{1}$$

   Note that $1 - \tan x \tan y \neq 0$, otherwise $\tan x + \tan y = 0 \Rightarrow 1 - \tan x \tan y = 1 + (\tan x)^2 > 0 \Rightarrow$ contradiction. Dividing (1) and using the formula $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ we get $\tan(x + y) = 1 \Rightarrow x + y = \dfrac{\pi}{4} + \pi k$, $k \in \mathbb{Z} \Rightarrow$ the minimal positive $x + y$ is equal to $\dfrac{\pi}{4}$.

8. In how many regions do n straight lines divide the plane if any two lines are not parallel and no three lines have a common point?

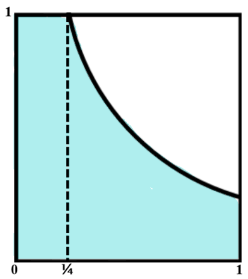
   Answer $\boxed{\dfrac{n^2+n+2}{2}}$.

   Solution Assume that n lines satisfying the conditions above divide the plane into a_n regions and draw one more line, so that the obtained $n + 1$ lines also satisfy the conditions above. The new line intersects all n previous lines in n distinct points, so that these intersection points divide the new line in $n + 1$ segments/rays. Each such segment divides one existing region into two, so that the number of regions after drawing the new line is increased ny $n + 1$, i.e. $a_{n+1} = a_n + n + 1$. Obviously $a_1 = 2$ so $a_n = 2 + 2 + 3 + \ldots + n = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}$.

9. Real numbers a, c are independently chosen uniformly at random from the open interval $(0, 1)$. What is the probability that the quadratic equation $ax^2 + x + c = 0$ has real roots?

Answer $\boxed{\frac{1}{4}(1 + \ln 4)}$.

Solution The quadratic equation $ax^2 + x + c = 0$ has real roots if and only if its discriminant $D = 1 - 4ac$ is nonnegative. Geometrically it means that the point (a, c) lies below or on the graph of the function $c = \frac{1}{4a}$. Therefore, since under our assumptions (a, c) also lies in the unit square $\{(a, c) \in \mathbb{R}^2 : 0 < a < 1, 0 < c < 1\}$, the desired probability is equal to the area of the region which is the part of this unit square that lies below the graph of $c = \frac{1}{4a}$. This area is equal $\dfrac{1}{4} + \displaystyle\int_{1/4}^{1} \frac{1}{4a} da = \frac{1}{4} + \frac{1}{4} \ln a|_{\frac{1}{4}}^{1} = \frac{1}{4} - \frac{1}{4} \ln \frac{1}{4} = \frac{1}{4}(1 + \ln 4)$ (see also the picture below).



10. Evaluate

$$\sqrt[3]{1\sqrt[3]{2\sqrt[3]{4\sqrt[3]{8\sqrt[3]{16\sqrt[3]{\cdots}}}}}}$$

Answer: $\boxed{\sqrt[4]{2}}$

Solution We rewrite this using fractional exponents: $1^{\frac{1}{3}} \cdot 2^{\frac{1}{9}} \cdot 4^{\frac{1}{27}} \cdots$. We can then rewrite all the terms as powers of 2. The expression then becomes $2^{\frac{0}{3}} \cdot 2^{\frac{1}{9}} \cdot 2^{\frac{2}{27}} \cdot 2^{\frac{3}{81}} \cdots$, or $2^{\frac{0}{3}+\frac{1}{9}+\frac{2}{27}+\frac{3}{81}+\cdots}$. The infinite sum in the exponent is equal to $\displaystyle\sum_{k=0}^{\infty} \frac{k}{3^{k+1}}$. Note that for any number x, such that $|x| < 1$, we have

$$\sum_{k=0}^{\infty} kx^{k+1} = \frac{x^2}{(1-x)^2}. \tag{2}$$

Applying this formula with $x = \dfrac{1}{3}$ we get that $\displaystyle\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} = \frac{\frac{1}{9}}{\frac{4}{9}} = \frac{1}{4}$, which implies our answer.

A quick way to prove (2) is to differentiate both parts of the formula for the infinite sum of geometric progression, $\sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$ and multiply the result by x^2. However, the fact that one can differentiate a power series term by term has to be justified and the tools for this are not of a school level.

A more elementary (but longer) way to prove it is as follows: for any n, using the formula for geometric progression several times , we have

$$\sum_{k=0}^{n} kx^{k+1} = \sum_{k=1}^{n}\sum_{i=k}^{n} x^{i+1} = \sum_{k=1}^{n} x^{k+1}\sum_{i=0}^{n-k} x^{i} = \sum_{k=1}^{n} x^{k+1}\frac{1-x^{n-k+1}}{1-x} = \frac{1}{1-x}\sum_{k=1}^{n} x^{k+1} - \frac{nx^{n+2}}{1-x}$$

Then, if $|x| < 1$, the second term approaches 0 as $n \to \infty$, while for the first term

$$\frac{1}{1-x}\sum_{k=1}^{n} x^{k+1} = \frac{x^2}{1-x}\sum_{k=0}^{n-1} x^{k} \to \frac{x^2}{(1-x)^2} \quad \text{as } n \to \infty,$$

which implies (2).

11. John placed 2017 letters A,B, and C in a row in such a way that at least one other letter lies between any two A's, at least two other letters lie between two B's , and at least three other letters lie between any two C's. What is the maximal possible number of C's in this row of letters?

    Answer 505.

    Solution Let us prove that among any 4 consecutive letters there is exactly one C. By assumptions, 4 consecutive numbers cannot have more than one C's, more than two B's, and more than two A's. Therefore , if this quadruple of letters does not contain C, then it must contain two B's. In this case between these two B's there are two letters and one of them is C, because two consecutive A are not allowed. So, we got a contradiction and among any 4 consecutive letters there is exactly one C. Since $2017 = 4 \times 504 + 1$, then the numbers of C's can be either 504 or 505. The 505 C's can be realized by placing C on each position having remainder $1 \mod 4$, placing A in all position having remainder 0 or $2 \mod 4$, and placing B in all position having remainder $3 \mod 4$, i.e. $CABACABA \ldots CABAC$

12. Among all $a > 1$ find the value of a for which the equation $a^x = \log_a x$ has exactly one solution.

    Answer $\boxed{e^{e^{-1}}}$

    Solution

    We use that the function $\log_a x$ is inverse to the function a^x. If x is the unique solution and $y = a^x = \log_a x$, then $\log_a y = x = a^y$, i.e. y is also a solution of the same equation. Since the solution is unique, we get that $y = x$. Therefore $x = a^x$. Note that any solution of the equation $x = a^x$ will also satisfy the original equation (again because in the original equation the function on one side is the inverse of the function on the other side). This means that also the equation

    $$a^x = x \tag{3}$$

    must have unique solution. Since the function $f(x) = a^x$ is convex, the solution of (3) is unique if and only if the graph of f is tangent to the line $y = x$ at their point of intersection. Therefore if x_0 is this unique solution, then the slope of f at x_0 must be equal to 1, i.e.

    $$(\ln a)a^{x_0} = 1 \tag{4}$$

    and also we know that $a^{x_0} = x_0$, so $(\ln a)x_0 = 1 \Rightarrow x_0 = \frac{1}{\ln a}$. Plugging this back to (4) and using that $a^{\frac{1}{\ln a}} = e$ we get that $\ln a = e^{-1}$ and so $a = e^{e^{-1}}$.

13. A sequence $x_1, x_2, x_3, \ldots$ is called periodic if there exists a positive integer T such that $x_{n+T} = x_n$ for any n and the minimal T satisfying this property is called the period of this sequence. Suppose that two sequences with period 7 and 13 coincide up to their kth elements (including the kth elements). What is the maximal possible k?

    Answer $\boxed{18}$.

    Solution Suppose that the first k elements of two sequences coincide and equal to $a_1, \ldots a_k$. Assume that k is sufficiently large so that all indices appearing in the subsequent relation do not exceed k. We get $a_7 = a_{7+7} = a_{14} = a_{14-13} = a_1 = a_{1+2\cdot 7} = a_{15} = a_{15-13} = a_2 = a_{1+2\cdot 7} = a_{16} = a_{16-13} = a_3 = a_{3+2\cdot 7} = a_{17} = a_{17-13} = a_4 = a_{4+2\cdot 7} = a_{18} = a_{18-13} = a_5 = a_{5+2\cdot 7} = a_{19} = a_{19-13} = a_6$, So $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7$. Therefore the first sequence is constant, i.e. its period is 1 and not 7. This shows that k cannot exceed 18. To show that the maximal possible k is 18 it is enough to give an example: consider the sequence of period 7 with the first seven elements $0, 0, 0, 0, 0, 1, 0$ and the sequence of period 13 with the 13 first elements $0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1$. Then the first 18 elements $0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0$ are common for the two sequences.

14. Regular hexagon ψ has side length 1. What is the area of the union of all regular hexagons Ψ in the same plane as ψ such that each vertex of ψ lies on a different side of Ψ?

    Answer: $\boxed{\dfrac{2\pi}{3} + \sqrt{3}}$

    Solution Consider a single side of ψ. We know that its endpoints lie on two adjacent sides of Ψ and that the angle at the vertex between these sides is $120°$. The set of all points thas could satisfy this is the circular arc with endpoints at the two endpoints of the side of ψ and measure $120°$. Using 30-60-90 triangles, we see that the radius of this particular arc is $\frac{\sqrt{3}}{3}$. Thus the area of the entire circle is $\frac{\pi}{3}$, the area of the $120°$ "slice" of the circle is $\frac{\pi}{9}$, and the area inside the arc but outside the original hexagon ψ is $\frac{\pi}{9} - \frac{\sqrt{3}}{12}$. (Because the area of the triangle with vertices at the endpoints of the original segment and the center of the circle is $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{6}$.) Now, there is an identical arc on each side of the hexagon, and the area of the union is the combined area inside all six of these arcs, plus the area of ψ. This is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$, or $\boxed{\dfrac{2\pi}{3} + \sqrt{3}}$.

15. Evaluate the following integral

$$\int_0^\pi \Big(\sin x + \sin(2x) + \sin(3x) + \ldots + \sin(2016x)\Big)^2 \, dx.$$

Answer $\boxed{1008\pi}$.

Solution

$$\int_0^\pi \Big(\sum_{n=1}^{2016} \sin(nx)\Big)^2 \, dx = \sum_{n=1}^{2016} \sum_{m=1}^{2016} \int_0^\pi \sin(nx)\sin(mx)\, dx \tag{5}$$

Note that

$$\int_0^\pi \sin(nx)\sin(mx)\, dx = \frac{1}{2}\int_0^\pi (\cos(n-m)x - \cos(n+m)x)\, dx.$$

Furthermore, $\displaystyle\int_0^\pi \cos kx\,dx = 0$ if k is a nonzero integer and it is equal to π if $k = 0$, Therefore,

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = 0 \text{ for any two distinct positive integers } n, m$$

and $\displaystyle\int_0^\pi \sin^2(nx)\,dx = \frac{\pi}{2}$. Hence, continuing the equality in (5), we get that

$$\int_0^\pi \Big(\sum_{n=1}^{2016} \sin(nx)\Big)^2 dx = \sum_{n=1}^{2016} \int_0^\pi \sin^2(nx)\,dx = 2016\frac{\pi}{2} = 1008\pi.$$

16. Assume that a function $f(x)$ is twice differentiable on the interval $[0,1]$, $f(0) = f(1) = 0$ and $|f''(x)| \le 1$ on $[0,1]$. Let $M(f)$ be the maximal value of f on the interval $[0,1]$. What is the maximal possible value of $M(f)$ among all functions f satisfying the properties above?

    Answer $\boxed{\frac{1}{8}}$.

    Solution The function $g(x) = \frac{1}{2}x(1-x)$ satisfies the conditions of the problem: $g(0) = g(1) = 0$ and $|g''(x)| = 1$ for any x, and $M(g) = g(\frac{1}{2}) = \frac{1}{8}$. Now prove that $\frac{1}{8}$ is the maximal possible value of $M(f)$ on the considered set of functions. Assume by contradiction that there exists a function f such that $f(0) = f(1) = 0$, $|f''(x)| \le 1$ for all $\in [0,1]$ and $M(f) > \frac{1}{8}$. Assume that f attains its maximum at some point $a \in [0,1]$. Let $h(x) = f(x) - \frac{f(a)}{g(a)}g(x)$. Obviously, $h(0) = h(1) = 0$. Since $g''(x) = -1$ and $f(a) > \frac{1}{8} \ge g(a) > 0$, , $h''(x) = f''(x) + \frac{f(a)}{g(a)} > 0$, which implies that h is concave up. Therefore, h has no more than two zeros, but it has at least three: at 0, a, and 1, a contradiction.

17. Find the positive integer k for which $A_k = \dfrac{10^k + 99^k}{k!}$ has the maximal value.

    Answer $\boxed{98}$

    Solution Let $B_k = \dfrac{10^k}{k!}$, $C_k = \dfrac{99^k}{k!}$. Then $A_k = B_k + C_k$, $\dfrac{B_{k+1}}{B_k} = \dfrac{10}{k+1}$, and $\dfrac{C_{k+1}}{C_k} = \dfrac{99}{k+1}$. Hence , if $k \le 10$ both sequences B_k and C_k are not decreasing (and C_k is strictly increasing), and if $k \ge 98$ both sequences are not increasing (and B_k is strictly decreasing), so that the maximum of A_k is attained for some k in the interval $[10, 98]$. Note that for $k \in [10, 97]$ we have $\frac{C_{k+1}}{C_k} \ge \frac{99}{98}$ and $\frac{B_k}{C_k} = \left(\frac{10}{99}\right)^k < \left(\frac{1}{9}\right)^{10} < \frac{1}{98}$. Therefore,

$$A_{k+1} - A_k = C_{k+1} - C_k + B_{k+1} - B_k \ge \frac{99}{98}C_k - C_k - B_k \ge \frac{1}{98}C_k - B_k > 0.$$

    Hence the sequence A_k is strictly increasing on $[10, 98]$ and A_k achieves its maximal value for $k = 98$.

18. What is the maximal number of factors of the form $\sin\frac{\pi n}{x}$ that can be erased on the left-hand side of the equation

$$\sin\frac{\pi}{x}\sin\frac{2\pi}{x}\ldots\sin\frac{2016\pi}{x} = 0 \tag{6}$$

so that the number of positive integer solutions will not change.

Answer $\boxed{1008}$

Solution The factor $\sin \dfrac{n\pi}{x}$ vanishes if and only if $\dfrac{n\pi}{x} = \pi k$, $k \in Z$, or, equivalently, $x = \dfrac{n}{k}$. This number is positive integer if and only if k is a (positive) divisor of n. Let $A_1 = \{1, 2, \ldots 1008\}$ and $A_2 = \{1009, 1010, \ldots 2016\}$. If $n \in A_2$, then the factor $\sin \dfrac{n\pi}{x}$ vanishes at $x = n$ and there is no other factor in the product (6) that is equal to zero at $x = n$, because $x = n$ is not a divisor of any other element of the set $\{1, 2, \ldots 2016\}$. Therefore, we cannot erase any factor of the form $\sin \dfrac{n\pi}{x}$ with $n \in A_2$. So, the number of factors that can be erased is at most 1008. On the other hand, if $n \in A_1$, then if $\sin \dfrac{n\pi}{x_0} = 0$ for some positive integer x_0, then x_0 is a divisor of some $N \in A_2$, therefore $\sin \dfrac{N\pi}{x_0} = 0$ as well. Hence, we can erase all 1008 factors $\sin \dfrac{n\pi}{x_0} = 0$ with $n \in A_1$ without changing the number of positive integer solutions of (6).

19. What is the 2016th digit after the decimal point of the number $(\sqrt{31} + \sqrt{32})^{2016}$?

Answer $\boxed{9}$

Let $k = (\sqrt{31} + \sqrt{32})^{2016} + (\sqrt{31} - \sqrt{32})^{2016}$. Binomial expansions of both terms reveal the fact that the odd powers cancel, and the terms with even powers are all integers, hence k is a positive integer. On the other hand, $|\sqrt{31} - \sqrt{32}| = \dfrac{1}{|\sqrt{31} + \sqrt{32}|} < \dfrac{1}{10}$, so $0 < (\sqrt{31} - \sqrt{32})^{2016} < 10^{-2016}$. Thus, the first 2016 digits after the decimal point of $(\sqrt{31} - \sqrt{32})^{2016}$ are 0's (but the fraction is nonzero), hence, since $(\sqrt{31} + \sqrt{32})^{2016} = k - (\sqrt{31} - \sqrt{32})^{2016}$, where k is integer, the first 2016 digits after the decimal point of $(\sqrt{31} + \sqrt{32})^{2016}$ are 9's.