CD Exam
Texas A&M High School Math Contest
October 21, 2017

All answers must be simplified, and if units are involved, be sure to include them.

1. Solve the equation $\sqrt{x+7} = x + 1$.

2. It can be shown that the number

$$A = \sqrt{|2\sqrt{7} - 8|} - \sqrt{8 + 2\sqrt{7}} + 2019$$

   is an integer. Find its value.

3. The atomic number of antimony is one less than four times the atomic number of aluminium. If twice the atomic number of aluminium is added to that of antimony, the result is the atomic number of the element iridium, which is 77. What is the atomic number of antimony?

4. Find the radius of the circle with the equation

$$2x^2 + 2y^2 + 4x + 24y + 73 = 0.$$

5. If n is a positive integer then the expression

$$A = \frac{3^{n+1} \cdot 4^n + 3^n \cdot 4^{n+1} + 12^{n+1}}{2^{n+1} \cdot 3^{n+1} - 5 \cdot 6^n}$$

   is also a positive integer. Find $\sqrt[n]{A/19}$.

6. To lay wall-to-wall carpeting in a rectangular shaped living room and dining room takes $612 \ ft^2$ of carpet. The living room is $3 \ ft$ longer than it is wide. The dining room floor is $6 \ ft$ wider than the width of the living room, and its length is twice the width of the living room. Find the width of the living room.

7. In the circle below, $CD = 14$, $AE = 3$, and $EB = 16$. Find $CE^2 + ED^2$.
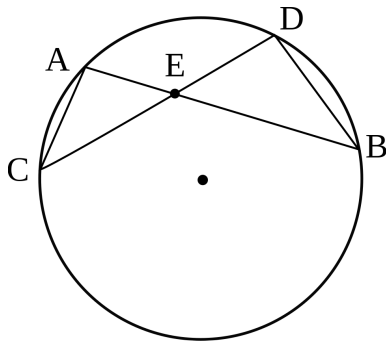
8. If $1072^x = 8$ and $67^y = 8$, find $\dfrac{1}{x} - \dfrac{1}{y}$.

9. Find the last digit of the number

$$A = 142^1 + 142^2 + 142^3 + \cdots + 142^{20}.$$

10. Find the sum of the squares of all solutions of the equation $|x^2 - 2x - 25| = 10$.

11. A community center plans to have a fund raising concert. There are 200 VIP tickets and 1000 general tickets. A VIP ticket costs $100 more than a general ticket. What should be the minimum price of a general ticket so that the funds raised would be at least $80,000?

12. How many positive integers n have the property that when $1,000,063$ is divided by n, the remainder is 63?

13. The diagram below shows a cup in the form of an inverted cone of height 18 cm and base radius 7.5 cm. It is filled with water to depth of 12 cm. Find the additional volume of water required to fill up the cup. Express your answer in terms of π.
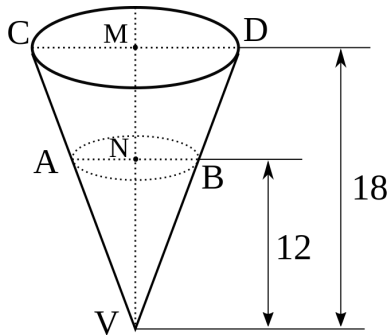
14. Find $3a + 5b$ if the polynomial $P(x) = 2ax^3 + 2bx^2 + 3x + 5$ is divisible by $x + 1$ and the sum of its coefficients is a positive and prime number.

15. Simplify
$$\frac{1^2}{1^4 + 1} + \frac{2^2 - 1}{2^4 + 2} + \frac{3^2 - 2}{3^4 + 3} + \ldots + \frac{1000^2 - 999}{1000^4 + 1000}.$$

16. Let $ABCD$ be a square and M and N be the midpoints of the sides BC and CD, respectively. Find $\sin(\angle MAN)$.

17. Consider the expression
$$E(x) = \frac{1}{x^3} + \frac{1}{x^2} + x^2 + x^3.$$
If a is a solution of the equation $x + \frac{1}{x} = 3$, find $E(a)$.

18. Let $\triangle ABC$ be an acute triangle and let D and E be such that BC is a perpendicular bisector of AD and AC is a perpendicular bisector of BE. Find $\angle ACB$ if the points D, C, and E are collinear.
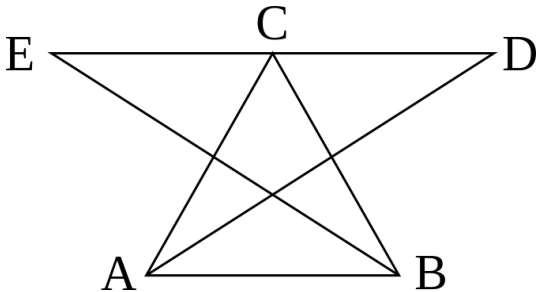
19. Find the minimum value of the expression
$$E(x, y) = 5(x^2 + y^2) - 2(-x + 4xy + 8y) + 36,$$
where x and y are real numbers.

20. In the right triangle ABC we have that $\angle A = 90°$ and D belongs to the side BC so that AD is perpendicular to BC. Find $\tan \angle ABC$ if the area of $\triangle ABC$ is $2AD^2$ and $\angle ABC > \angle ACB$.