DE EXAM

Texas A&M High School Math Contest

October 2017

Directions: If units are involved, include them in your answer.

1. Simplify $\left(a - \frac{1}{a}\right)\left(a^2 + 1 + \frac{1}{a^2}\right)$.

   Solution. $\left(a - \frac{1}{a}\right)\left(a^2 + 1 + \frac{1}{a^2}\right) = a^3 + a + \frac{1}{a} - a - \frac{1}{a} - \frac{1}{a^3} = a^3 - \frac{1}{a^3}$

   Answer: $a^3 - \frac{1}{a^3}$

2. Find the value $\log_b a^2 \cdot \log_a b^2$.

   Solution. $\log_b a^2 \cdot \log_a b^2 = 2\log_b a \cdot 2\log_a b = 2 \cdot 2 \log_b a \log_a b = 4$

   Answer: 4

3. If the radius of a circle is increased by 100% the area is increased by $a\%$. Find a.

   Solution. The area of a circle with radius $2r$ is $\pi(2r)^2 = 4\pi r^2$. So the difference is $4\pi r^2 - \pi r^2 = 3\pi r^2$ or the area increases by 300%.

   Answer: 300

4. Find the value of c so that the vertex of a parabola $y = x^2 - 8x + c$ is a point on the x-axis.

   Solution. Since $y = x^2 - 8x + c = (x-4)^2 + c - 16$ the vertex of the parabola is $(4, c-16)$.

   Answer: $c = 16$.

5. A regular hexagon is inscribed in a circle. Find the ratio of the length of a side of the hexagon to the length of the corresponding arc.

   Solution. If the circle has radius r, each fan of angle $\frac{\pi}{3}$ has the arc length $r\frac{\pi}{3}$. So the ratio is

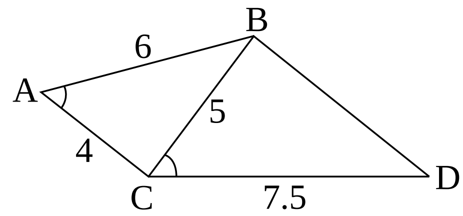
   $$\frac{r}{\frac{\pi r}{3}} = \frac{3}{\pi}.$$

   Answer: $\frac{3}{\pi}$

6. For every natural number n, the sum S_n of n terms of an arithmetic progression is $3n + 4n^2$. Find the k^{th} term.

   Solution. The k^{th} term is given by $S_k - S_{k-1} = 3k + 4k^2 - \left(3(k-1) + 4(k-1)^2\right) = 8k - 1$.

   Answer: $8k - 1$



7. As shown in the figure below, $\angle BAC = \angle BCD$, $AC = 4$ inches, $BC = 5$ inches, $AB = 6$ inches, and $CD = 7.5$ inches. Find BD.

Solution. The triangle $\triangle BAC$ is similar to the triangle $\triangle DCB$ since $\angle BAC = \angle DCB$ and $AC : CB = AB : CD = 4 : 5$. Thus

$$\frac{5}{BD} = \frac{4}{5} \text{ or } BD = \frac{25}{4}$$

Answer: $\frac{25}{4}$ inches

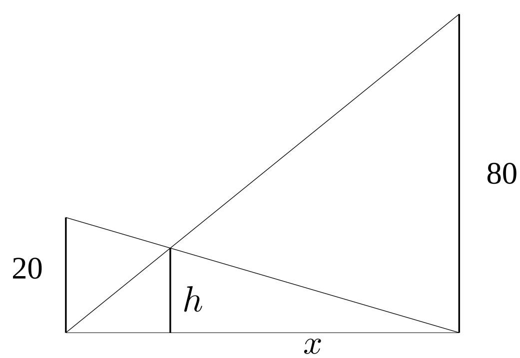
8. Suppose two poles $20''$ and $80''$ high are $100''$ apart. Find the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole.

Solution. Let h be the height of the intersection and x be the distance from the intersection to the longer pole. Comparing similar triangles we have

$$20 : 100 = h : x \text{ and } 80 : 100 = h : (100 - x).$$

Solving the above we have $h = 16$ and $x = 80$.

Answer: $16''$



9. Find x if $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$.

Solution. By the condition x satisfies $x = \sqrt{1 + x}$. Consequently x is a root of $x^2 - x - 1 = 0$. Since $x > 0$, $x = \frac{1 + \sqrt{5}}{2}$.

Answer: $\dfrac{1+\sqrt{5}}{2}$

10. Find the intersection of graphs $y = \log 2x$ and $y = 2\log x$.

    Solution. Solving the equation $\log 2x = 2\log x$ we have

    $$\log 2 + \log x = 2\log x \Rightarrow \log 2 = \log x \Rightarrow x = 2.$$

    Answer: $(2, 2\log 2)$ or $(2, \log 4)$.

11. If r and s are the roots of the equation $ax^2 + bx + c = 0$, express $\dfrac{1}{r^2} + \dfrac{1}{s^2}$ in terms of a, b and c.

    Solution. Since $r + s = -\dfrac{b}{a}$ and $rs = \dfrac{c}{a}$,

    $$\frac{1}{r^2} + \frac{1}{s^2} = \frac{r^2 + s^2}{r^2 s^2} = \frac{(r+s)^2 - 2rs}{(rs)^2} = \frac{\left(\dfrac{-b}{a}\right)^2 - \dfrac{2c}{a}}{\left(\dfrac{c}{a}\right)^2} = \frac{b^2 - 2ac}{c^2}$$

    Answer: $\dfrac{b^2 - 2ac}{c^2}$

12. Find the last digit of the number

    $$A = 142^1 + 142^2 + 142^3 + \cdots + 142^{20}.$$

    Solution. We can write A as

    $$\begin{aligned} A &= (142^1 + 142^2 + 142^3 + 142^4) + (142^5 + 142^6 + 142^7 + 142^8) \\ &\quad + \cdots + (142^{17} + 142^{18} + 142^{19} + 142^{20}) \\ &= (142^1 + 142^2 + 142^3 + 142^4) + 142^4(142^1 + 142^2 + 142^3 + 142^4) \\ &\quad + \cdots + 142^{16}(142^1 + 142^2 + 142^3 + 142^4). \end{aligned}$$

    The last digits of 142^1, 142^2, 142^3, and 142^4 are 2, 4, 8, and 6 respectively. Therefore the last digit of $142^1 + 142^2 + 142^3 + 142^4$ is 0 since $2 + 4 + 8 + 6 = 20$. We conclude that the last digit of A is 0.

    Answer: 0

13. How many integers satisfy the inequality $x^2 - x - 1980 < 0$?

    Solution. We have

    $$x^2 - x - 1980 < 0 \Leftrightarrow (x - 45)(x + 44) < 0$$

    The given inequality has solution $-44 < x < 45$. There are 88 integers in the interval $(-44, 45)$.

    Answer: 88

14. Find $\left(r^3 + \dfrac{1}{r^3}\right)^{2017}$ if $\left(r + \dfrac{1}{r}\right)^2 = 3$.

Solution. Since $\left(r + \dfrac{1}{r}\right)^2 = 3$, $r^2 + \dfrac{1}{r^2} = 1$ or $r^2 - 1 + \dfrac{1}{r^2} = 0$. Thus $\left(r^3 + \dfrac{1}{r^3}\right)^{2017} = 0$ since

$$r^3 + \frac{1}{r^3} = \left(r + \frac{1}{r}\right)\left(r^2 - 1 + \frac{1}{r^2}\right) = 0.$$

Answer: 0

15. It takes 30 minutes for a man to commute by his car. If he speeds up by 20% how long will it take?

Solution. If the speed increases by a factor of 1.2 then the time decreases by a factor of $\dfrac{1}{1.2} = \dfrac{5}{6}$ since the distance = speed×time.

Answer: 25 min

16. Simplify

$$\frac{1^2}{1^4+1} + \frac{2^2-1}{2^4+2} + \frac{3^2-2}{3^4+3} + \cdots + \frac{1000^2-999}{1000^4+1000}.$$

Solution. We notice that all terms are of the form

$$\begin{aligned}\frac{n^2-(n-1)}{n^4+n} &= \frac{n^2-n+1}{n(n^3+1)} = \frac{n^2-n+1}{n(n+1)(n^2-n+1)} \\ &= \frac{1}{n(n+1)} = \frac{(n+1)-1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\end{aligned}$$

Therefore, our expression is equal to

$$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{1000}-\frac{1}{1001}\right) = 1 - \frac{1}{1001} = \frac{1000}{1001}$$

Answer: $\dfrac{1000}{1001}$

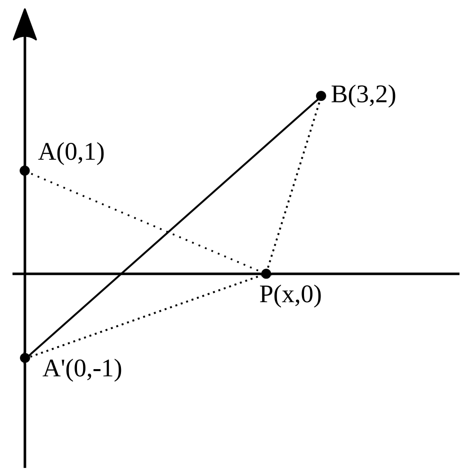
17. Given points $A = (0,1)$, $B = (3,2)$ and an arbitrary point P on the x-axis. Find the minimal value for the length $AP + PB$.

Solution. Let $A' = (0,-1)$ (See figure below). By the triangle inequality we have

$$AP + PB = A'P + PB \geq BA' = \sqrt{3^2+3^2} = 3\sqrt{2}.$$

Answer: $3\sqrt{2}$

18. Find the maximum of $-x - \dfrac{1}{x}$ for $x > 0$.



Solution. By the inequality of arithmetic and geometric means we have

$$x + \frac{1}{x} \geq 2\sqrt{1 \cdot \frac{1}{x}} = 2.$$

Answer: -2

19. Simplify $\cos 10^\circ + \cos 20^\circ + \cdots + \cos 180^\circ$.

Solution. Since $\cos x^\circ + \cos(180^\circ - x^\circ) = 0$ for all $0 \leq x \leq 180$, the sum is

$$\begin{aligned}
&\cos 10^\circ + \cos 20^\circ + \cdots + \cos 180^\circ \\
=&(\cos 10^\circ + \cos 170^\circ) + (\cos 20^\circ + \cos 160^\circ) + \cdots + (\cos 80^\circ + \cos 100^\circ) + \cos 90^\circ + \cos 180^\circ \\
=&\cos 180^\circ = -1
\end{aligned}$$

Answer: -1

20. Suppose the hour hand and minute hand of a clock make an angle of 135°. Assuming the hours and minutes are integers, what is the time? Write your answer in the form $h : m$, where h is hours and m is minutes.

Solution. Suppose the clock reads $h : m$ now ($0 \leq h \leq 11$, $0 \leq m \leq 59$). The angle between the hour hand and 12 o'clock is $30h + \frac{30}{60}m$. And the angle between the minute hand and 12 o'clock is $\frac{360}{60}m$. The difference between two angles must be 135°, and so we have

$$30h + \frac{30}{60}m - \frac{360}{60}m = \pm 135^\circ.$$

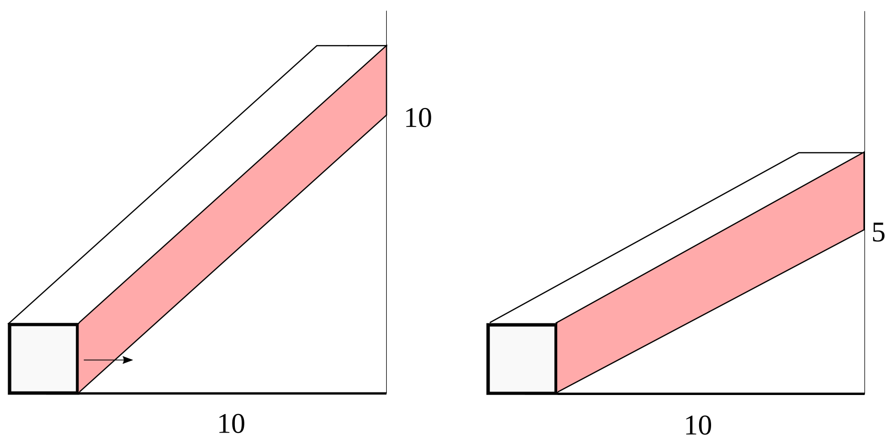
Since $60h - 11m = 270$ or $60h - 11m = -270$, m is a multiple of 30. The only integer solutions for h and m are

$$\begin{cases} h = 1 \\ m = 30 \end{cases} \quad \text{and} \quad \begin{cases} h = 10 \\ m = 30. \end{cases}$$

Answer: 1:30 or 10:30.

21. Suppose a cube has side 1 foot and moves 10 ft along a straight path. Assume that it rains and all rain drops of the same size fall vertically with the same speed of 1 ft/s. Assume further that they are small enough so that the number of rain drops is the same in each cubic feet. The amount of rain that the cube gets (on the front and top faces) when it travels with speeds of 2 ft/s is a% of the amount when traveling 1 ft/s. Find a.

Solution. Consider the location of rain drops which collect on the front face and the top face while the cube moves. The rain drops which collect on the front face belong to the space determined by the shaded parallelogram in the figure. The left figure is for the speed 1 ft/s and the right figure is for the speed 2 ft/s. Since the cube tavels 10 feet both shaded parallelograms have the base 1 foot and the height 10 feet. On the other hand the rain drops that fall onto the top face lie in the space determined by the white parallelograms. The left white parallelogram has the height 10 feet since the speed of cube = 1 ft/s = the speed of rain drops, while the right one has the height 5 feet. Let R denote the amount of rain in each cubic feet. The amount A_1 of



rain that the cube takes with speed 1ft/s is

$$A_1 = R(1 \cdot 1 \cdot 10 + 1 \cdot 1 \cdot 10) = 20R$$

while the amount A_2 of rain that it takes with speed 2ft/s is

$$A_2 = R(1 \cdot 1 \cdot 10 + 1 \cdot 1 \cdot 5) = 15R.$$

Thus $A_2 = 0.75A_1$.

Answer: $a = 75$.

22. Solve the equation $x + \sqrt{25 - 6x} - 3 = 0$.

Solution. The given equation becomes

$$x - 3 = -\sqrt{25 - 6x} \Rightarrow (x-3)^2 = 25 - 6x \Rightarrow x^2 - 16 = 0 \Rightarrow x = \pm 4$$

However only $x = -4$ satisfies the given equation.

Answer: $x = -4$