EF Exam
Texas A&M High School Math Contest
October, 2017

Answers should include units when appropriate.

1. A drawer contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. What is the smallest number of socks that must be selected (without looking) to guarantee that the selection contains at least 10 pairs (a pair is two socks of the same color).

2. Alice rolls a fair regular octahedral die marked with the numbers 1 through 8. Then Bob rolls a fair six-sided die. What is the probability that the product of two rolls is a multiple of 3?

3. Find the minimum value of $\sqrt{x^2 + y^2}$ when $5x + 12y = 60$.

4. Find the number of solutions in positive integers of $2x + 3y = 2017$.

5. An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge (the edge that has no common endpoint with the first edge). What is the length of the shortest such trip?

6. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find

$$\sum_{n=1}^{1024} \lfloor \log_2 n \rfloor.$$

7. Find all x in the interal $[0, \pi]$ such that $\sin^8 x + \cos^8 x = \frac{41}{128}$.

8. If a and b are positive real numbers such that each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then what is the smallest possible value of $a + b$?

9. For which of the numbers $n = 2017$ or $n = 2016$ is the polynomial $x^{2n} + (x + 1)^{2n} + 1$ divisible by $x^2 + x + 1$?

10. Find all pairs of integers (x, y) such that $x^3 + y^3 = 91$.

11. Point D is on side CB of triangle ABC. If $\angle CAD = \angle DAB = 60°$, $AC = 1$, and $AB = 2$, find AD.

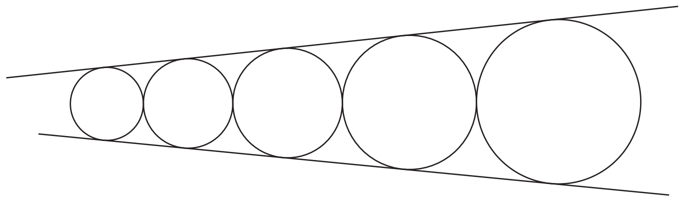
12. Evaluate the integral $\int_{-1}^{15} \frac{dx}{\sqrt{x+10} - \sqrt{x+1}}$.

13. Positive integers a, b, c are such that $a < b < c$, and the system

$$\begin{cases} 2x + y &= 2017 \\ |x - a| + |x - b| + |x - c| &= y \end{cases}$$

has exactly one solution. What is the smallest possible value of c?

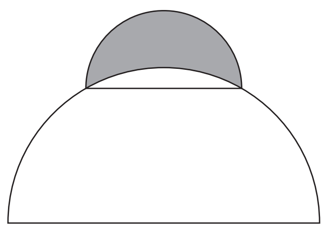
14. Five circles are tangent to one another consecutively and to the lines L_1 and L_2. Find the radius of the middle circle if the radius of the largest circle is 18 and the radius of the smallest one is 8.



15. Let $f(x) = x^3 - 3x + 1$. Find the number of distinct real roots of the polynomial $f(f(x))$.

16. Let A_1 and C_1 be points on the sides BC and AB, respectively, of the triangle $\triangle ABC$ such that AA_1 and CC_1 are perpendicular to $\overline{BC}$ and $\overline{AB}$, respectively. If $AA_1 \geq BC$ and $CC_1 \geq AB$, what are possible measures of the angles of the triangle $\triangle ABC$?

17. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. What is the area of the shaded area inside the smaller semicircle and outside the larger semicircle?



18. Let $f(x)$ be a function defined on the non-negative real numbers and taking non-negative real values, such that:

(i) $f(xf(y))f(y) = f(x+y)$ for all $x, y \geq 0$;

(ii) $f(2) = 0$;

(iii) $f(x) \neq 0$ for all $0 \leq x < 2$.

Find $f(\sqrt{2}) + f(\pi)$.

19. Evaluate the product

$$(\sqrt{3} + \tan 1^\circ)(\sqrt{3} + \tan 2^\circ) \cdots (\sqrt{3} + \tan 29^\circ).$$

20. Let $E(n)$ be the largest integer k such that 5^k divides $2^2 3^3 4^4 \cdots n^n$. Find $\lim_{n\to\infty} \frac{E(n)}{n^2}$.

21. Evaluate, for every positive integer n, the integral

$$\int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\, dx.$$