CD Exam
Texas A&M High School Math Contest
October 20, 2018

1. For any positive integer n let $D(n)$ denote the sum of its digits (in decimal notation). Find all integers n such that $n + 3D(n) = 2018$.

2. A 5×5 square drawn on the square grid is then cut into several rectangles of distinct areas (all cuts go along the grid lines). What is the maximal possible number of pieces in such a partition?

3. How many ways there are to change a \$20 bill into smaller bills? (You can use \$1, \$2, \$5 and \$10 bills; no coins are allowed.)

4. Given an equilateral triangle ABC with side 1, one chooses a point A_1 on the side BC, a point B_1 on the side AC, and a point C_1 on the side AB. What is the maximal possible perimeter of the triangle $A_1B_1C_1$?

5. Solve for x the equation $\big|3 - 2|x|\big| = x + 1$.

6. Find the last digit of the number $2018^{2017^{2016}}$.

7. Consider a right triangle ABC with $\angle A = 90^o$. Let D be the midpoint of the side BC and E be a point on the side AB such that $\angle AEC = \angle BED$. Find the ratio $\dfrac{|AE|}{|EB|}$.

8. The equation $x^4 + 4x^3 - 3x^2 + 4x + 1 = 0$ has two real solutions. Find their sum.

9. A positive integer m can be represented as $2^4p_1p_2p_3$, where p_1, p_2, p_3 are some odd prime numbers (not necessarily distinct). The integer $m+100$ can be represented as $5q_1q_2q_3$, where q_1, q_2, q_3 are prime numbers different from 5 (not necessarily distinct). The integer $m + 200$ can be represented as $23r_1r_2r_3r_4$, where r_1, r_2, r_3, r_4 are prime numbers different from 23 (not necessarily distinct). Find m.

10. Solve for x the inequality $\sqrt{x-1} \le 4 - 3x - x^2$.

11. Three points A, B and C divide a circle of radius 1 into three arcs of equal length. We draw three more circles, each intersecting the original circle at two of the three points. All circles intersect at right angles. Find the area of the curvilinear triangle ABC bounded by new circles.

12. Find all triples (x, y, z) satisfying the system

$$\begin{cases} xyz = 1, \\ x^4yz^2 = 4, \\ x^{10}yz^6 = 16. \end{cases}$$

13. Find all pairs of integers (x, y) satisfying the equation $2^x - 3^y = 1$.

14. Solve for x the equation $\sqrt{6+\sqrt{6+\sqrt{6+x}}}=x$.

15. Let ABC be an isosceles triangle with the base AB. Let AD be the median and AE be the angle bisector of this triangle. Find the length of the leg AC if $|AB|=5$ and $|DE|=6$.

16. Solve for x the equation $\log_2 x\cdot\log_2(x-2)+1=\log_2(x^2-2x)$.

17. Find all polynomials $P(x)$ such that $(P(x))^2=P(x-1)P(x+1)$ for all x.

18. In an acute triangle ABC with the median AD and the altitude AE, the point F at which the inscribed circle touches the side BC is the midpoint of the segment DE. Find the perimeter of the triangle ABC if $|BC|=1$.

19. Evaluate the sum

$$\frac{1}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\frac{3}{4\cdot 5\cdot 6}+\cdots+\frac{2015}{2016\cdot 2017\cdot 2018}.$$

20. Solve for x the equation $\sqrt[3]{2x-1}-\sqrt[3]{x-1}=1$.

21. Let $a_1,a_2,\ldots,a_{2018}$ be the numbers $1,2,\ldots,2018$ written in some order. What is the maximal possible value of $a_1a_2+a_3a_4+\cdots+a_{2017}a_{2018}$? The following equalities may (or may not) help to calculate the value:

$$\begin{gathered}1+2+3+4+\cdots+2017+2018=2037171,\\ -1^2+2^2-3^2+4^2-\cdots-2017^2+2018^2=2037171,\\ 1^2+2^2+3^2+4^2+\cdots+2017^2+2018^2=2741353109.\end{gathered}$$

22. A sequence of real numbers $a_1,a_2,a_3,\ldots$ is built recursively using the rule $a_{n+2}=a_na_{n+1}+1$ for $n=1,2,\ldots$ The sequence happens to be periodic, that is, $a_{n+k}=a_n$ for some $k\geq 1$ and all n. Let s_n be the sign of the number a_n (s_n is $+$ if $a_n>0$ and $-$ if $a_n<0$; for $a_n=0$ it can be either $+$ or $-$). Assuming s_1 is $+$, find all possible sequences $s_1,s_2,s_3,\ldots$ In each case, describe the sequence of signs completely or list at least the first 10 terms. (Hint: since the sequence $a_1,a_2,a_3,\ldots$ is periodic, some simple combinations of signs cannot occur in $s_1,s_2,s_3,\ldots$)