DE Exam
Texas A&M High School Math Contest
October 20, 2018

All answers must be simplified, and if units are involved, be sure to include them.

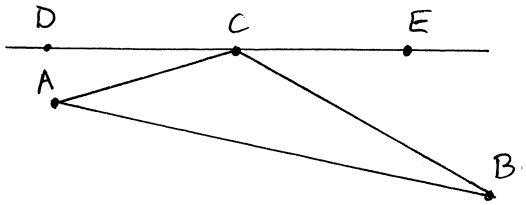
1. Solve the equation $4^{x-3} - 8^{x+5} = 0$.

2. Find the value of $\dfrac{y}{z}$ if $3wz + 4xy - 2wy - 6xz = 0$, $w \neq 2x$ and $z \neq 0$.

3. If $\log x + \log y = \dfrac{29}{10}$ and $\log x \log y = 1$ find the value of
$$\log_x y + \log_y x.$$

4. Let x be a real number and y be a positive integer such that $x > 1$ and $\dfrac{x}{3} = \dfrac{5x+1}{3y+2}$. Find y.

5. In the figure below we have $AC = 2$, $BC = 3$, $\angle DCA = 15°$, and $\angle ECB = 30°$. Find AB.



6. The probability that a worker with occupational exposure to dust contracts a lung disease is $\dfrac{1}{6}$. Three such workers are checked at random. Find the probability that at least one of them contracted a lung disease.

7. Find the value of $\tan 1° \tan 2° \tan 3° \cdots \tan 88° \tan 89°$.

8. Find xy, where x and y satisfy the system of equations
$$\begin{cases} \dfrac{1}{x-2} + \dfrac{1}{y} & = 6 \\ 4x + 47y - 22xy & = 8. \end{cases}$$

9. Find the real number k such that the equation $|x^2 - 2x - 8| = k$ has exactly three real distinct solutions.

10. Find the coefficient of x^2 in the expansion of $(2-x)^6(1+3x)^7$.

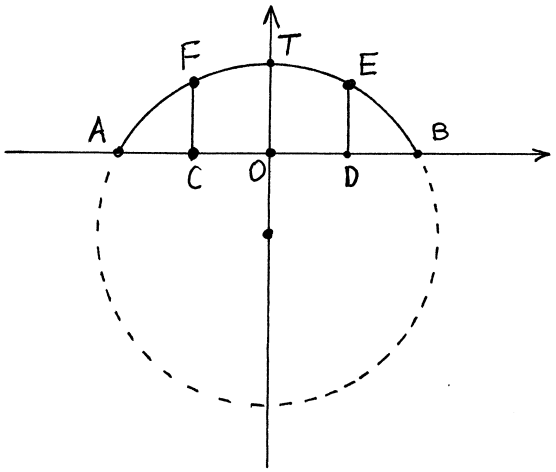
11. Determine the sum of all integers n such that the number $n^2 + 9n + 14$ is the square of another integer.

12. Find the maximum value of the expression $(2n^2 + 3n)\sqrt{3} - (3n^2 + 2n)\sqrt{2}$, where n is an integer.

13. Let $P(x)$ be a polynomial of degree at least two such that the remainders for the division of $P(x)$ by $x-3$ and $x+5$ are 5 and -11, respectively. Find the remainder of the division of $P(x)$ by $x^2 + 2x - 15$.

14. Simplify the fraction
$$\frac{27n^3 + 6n^2 - 37n + 4}{27n^3 - 21n^2 - 70n + 8}$$
and then find its value for $n = 56789$.

15. Consider cartesian coordinates with the origin at the point O and axes OB and OT. The diagram below shows the arch $AFTEB$ of a stone bridge. The bridge forms an arc of a circle and length AB forms a chord of the circle. AB is 24 feet and the top of the bridge T is 3 feet vertically above AB. C and D are midpoints of OA and OB. CF and DE are two vertical pillars supporting the arch. Find the height of the pillar DE.



16. Find the value of $\log_2(x_1x_2)$, where x_1 and x_2 are the solutions of the equation

$$\log_2 x^{\sqrt{5}+1} + \log_x 4^{\sqrt{5}+1} = \log_2(16x^3) - \log_x 16.$$

17. Consider the triangle ABC in which the angle bisector of $\angle A$ intersects side BC at a point M and the angle bisector of $\angle B$ intersects side AC at a point N. Let O be the intersection point between AM and BN. We know that $\dfrac{AO}{OM} = \sqrt{3}$ and $\dfrac{ON}{BO} = \sqrt{3} - 1$. Find $\angle C$.

18. Find the distance from the center to the foci of the hyperbola with vertices $(5, -6)$ and $(5, 6)$, passing through the point $(0, 9)$.

19. Find $\cot^2 36^\circ \cot^2 72^\circ$.

20. Find the minimum value of the function

$$f(x) = 1 \cdot |x - 1| + 2 \cdot |x - 2| + 3 \cdot |x - 3| + \cdots + 20 \cdot |x - 20|.$$