EF EXAM Solutions
Texas A&M High School Math Contest
October 20, 2018

1. Let N be the product of all numbers that appear on the 10×10 multiplication chart, i.e., the numbers of the form pq, where $1 \le p, q \le 10$. What is the largest number m such that $\sqrt[m]{N}$ is an integer?

Answer. 20.

Solution. We have

$$N = \prod_{p,q=1}^{10} pq = \prod_{q=1}^{10}\prod_{p=1}^{10} pq = \prod_{q=1}^{10} q^{10}(10!) = (10!)^{10}(10!)^{10} = (10!)^{20}.$$

The highest power of the prime number 7 that divides 10! is 7^1, so the largest number m such that $\sqrt[m]{(10!)^{20}}$ is an integer is 20.

2. How many real solutions does the following equation have?

$$(x+1)^{2018} + (x+1)^{2017}(x-2) + (x+1)^{2016}(x-2)^2 + \cdots + (x+1)(x-2)^{2017} + (x-2)^{2018} = 0$$

Answer. 0.

Solution. Using the identity

$$b^n - a^n = (b-a) \cdot \sum_{k=0}^{n-1} b^{n-1-k}a^k$$

for $n = 2019, b = x+1, a = x-2$, the given equation leads to

$$\frac{(x+1)^{2019} - (x-2)^{2019}}{(x+1)-(x-2)} = 0,$$

which implies, in real numbers, the absurd equation $x+1 = x-2$ due to the fact that 2019 is odd. (Note that there are 2018 complex solutions, though.)

3. Let f be a continuous function on $[0, 2018]$ such that $f(x)f(2018-x) = 1$, for all $x \in [0, 2018]$. Evaluate

$$\int_0^{2018} \frac{dx}{1+f(x)}.$$

Answer. 1009.

Solution. Let

$$I = \int_0^{2018} \frac{dx}{1+f(x)}.$$

Then substitution $u = 2018 - x$ gives

$$I = \int_0^{2018} \frac{dx}{1+f(2018-x)} = \int_0^{2018} \frac{dx}{1+\frac{1}{f(x)}} = \int_0^{2018} \frac{f(x)\,dx}{1+f(x)}.$$

In particular,

$$2I = I + I = \int_0^{2018} \frac{dx}{1+f(x)} + \int_0^{2018} \frac{f(x)\,dx}{1+f(x)} = \int_0^{2018} dx = 2018,$$

hence $I = 2018/2 = 1009$.

4. What is the number of natural numbers n with the property that $[\frac{n^2}{3}]$ is a prime number? Here, $[x]$ denotes the greatest integer that is not larger than x.

   Answer. 2.

   Solution. Using congruence mod 3, there are three cases: 1) $n = 3k$, so $[n^2/3] = 3k^2$, which is not a prime, unless $k = 1$ which happens when $n = 3$. 2) $n = 3k-2$, so $[n^2/3] = [3k^2-4k+1+1/3] = (3k-1)(k-1)$, which is not a prime, unless $k = 2$ which happens when $n = 4$. 3) $n = 3k-1$, so $[n^2/3] = [3k^2-2k+1/3] = k(3k-2)$, which is never a prime. So the only solutions are $n = 3, 4$.

5. What is the coefficient of x^5 in the expansion of the following polynomial?

$$(1 + 2x + 3x^2 + 4x^3 + \cdots + 2018\,x^{2017})^2(1 + x^4 + x^8)^2$$

   Answer. 64.

   Solution. Writing the powers of x up to x^5 in the expansions of the expression we get

$$(1 + x^4 + x^8)^2 = 1 + 2x^4 + \ldots$$

   and doing the same for the first term gives us

$$(1 + 2x + 3x^2 + 4x^3 + \cdots + 2018\,x^{2017})^2 = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + \ldots$$

   The only possibilities to get x^5 from the product of the above expression is through $1(56x^5)+2x^4(4x) = 64x^5$.

   As a second solution, note that according to the expansion of $(1 + x^4 + x^8)^2$, we only need to know the coefficients x and x^5 in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \cdots + 2018\,x^{2017})^2$, which shares the same information with the infinite series

$$f(x) = \left(\sum_{k=0}^{\infty} kx^{k-1}\right)^2 = \left(\frac{d}{dx}\sum_{k=0}^{\infty} x^k\right)^2 = \left(\frac{d}{dx}(1-x)^{-1}\right)^2 = (1-x)^{-4}$$

   for $-1 < x < 1$. By Taylor's formula, the coefficient of x must be $f'(0) = 4$ and coefficient of x^5 is

$$\frac{f^{(5)}(0)}{5!} = \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5!} = 56.$$

6. Consider the function $f(x, y) = y^2 - x^2 - 2xy + 2x + 1$. Jack and Janet play the following game: First, Jack plugs in a value for x, and then Janet plugs in a value for y. The value of the function will be considered as Jack's score. If Janet plays against Jack, what is the maximum score Jack can gain?

Answer. $3/2 = 1.5$.

Solution. Janet plays well if she minimize the function for the y-values given the value of x Jack has plugged in. Therefore, the solution to the problem is obtained by solving the optimization problem

$$\max_x \min_y f(x, y).$$

But $f(x, y)$ is a quadratic function in y for each fixed value of x, and the coefficient of y^2 is positive, so the function is concave up, with minimum at $y = -(-2x)/2(1) = x$, hence

$$\min_y f(x, y) = f(x, x) = -2x^2 + 2x + 1,$$

which is a quadratic in x that is concave down, with a maximum at $x = -2/2(-2) = 1/2$, so

$$\max_x \min_y f(x, y) = -2(1/2)^2 + 2(1/2) + 1 = 3/2.$$

7. We say that a natural number greater than one has property S if the sum of any of its two distinct divisors is divisible by 7. How many numbers with property S are less than 100?

   Answer. 4.

   Solution. A number n with such property cannot be composite: otherwise, it can be written as $n = ab$, with $a, b \geq 2$, and the property implies that

   $$7 \mid 1 + a,\ 7 \mid 1 + b,\ 7 \mid 1 + ab.$$

   The above implies

   $$7 \mid \{(1 + a)(1 + b) - (1 + ab)\} \Rightarrow 7 \mid (a + b)$$

   and

   $$7 \mid (a + 1) + (b + 1) \Rightarrow 7 \mid (a + b + 2),$$

   hence $7 \mid 2$, which is absurd. The only prime numbers p with property S are those satisfying $7 \mid p + 1$, which upon checking we have $p \in \{13, 41, 83, 97\}$.

8. Consider the parabola $y = x^2 - 2ax + 1$ and the line $y = 2b(a - x)$. Let A be the set of points $(a, b) \in \mathbb{R}^2$ such that the line and the parabola defined above do not intersect. Find the area of A as a region in $\mathbb{R}^2$.

   Answer. π.

   Solution. The two curves do not intersect if the quadratic equation

   $$x^2 - 2ax + 1 - 2b(a - x) = 0$$

   has negative discriminant, which implies

   $$(b - a)^2 - (1 - 2ab) = b^2 + a^2 - 2ab + 2ab - 1 = a^2 + b^2 - 1 < 0.$$

   This is a disk of radius 1 in the ab-plane, with area π.

9. For any natural number n, let $p(n)$ be the product of the digits in the decimal expansion of n. Find $p(1) + p(2) + p(3) + \cdots + p(999)$.

Answer. 93,195.

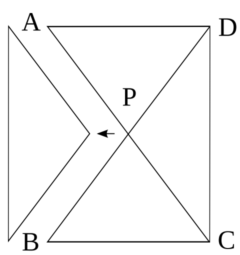
Solution. Note that based on the number of digits we have

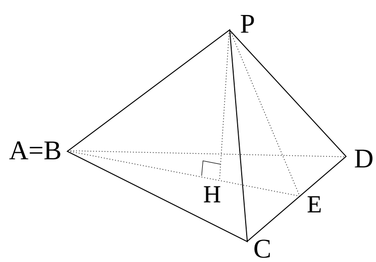
$$\begin{aligned}\sum_{n=1}^{9} p(n) &= \sum_{i=1}^{9} i = 45,\\ \sum_{n=10}^{99} p(n) &= \sum_{i=1}^{9}\sum_{j=0}^{9} (i \times j) = \left(\sum_{i=1}^{9} i\right)\left(\sum_{j=0}^{9} j\right) = \left(\sum_{i=1}^{9} i\right)^2 = 45^2,\\ \sum_{n=100}^{999} p(n) &= \sum_{i=1}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9} (i \times j \times k) = \left(\sum_{i=1}^{9} i\right)^3 = 45^3.\end{aligned}$$

Therefore

$$\sum_{n=1}^{999} p(n) = 45 + 45^2 + 45^3 = 93195.$$

10. Consider a rectangular paper $ABCD$ with $AB = 8$, $AD = 6$ and a point P, the intersection of two diagonals. Remove the triangle $\triangle PAB$, and then fold PC and PD so that PA and PB are identified. Find the volume of the tetrahedron determined by the resulting piece of paper.





Answer. $\dfrac{16\sqrt{11}}{3}$.

Solution. After identifying PA and PB we have a tetrahedron with the base triangle $\triangle ACD$. By the Pythagorean theorem, $AE = 2\sqrt{5}$ and $PE = 3$. Let $\angle PEA = \alpha$. The law of cosine applied to $\triangle PAE$ yields

$$\cos\alpha = \frac{9 + 20 - 25}{2 \cdot 3 \cdot 2\sqrt{5}} = \frac{1}{3\sqrt{5}}.$$

So $\sin\alpha = \sqrt{1 - \frac{1}{45}} = \frac{2\sqrt{11}}{3\sqrt{5}}$ implies

$$PH = PE \cdot \sin\alpha = \frac{2\sqrt{11}}{\sqrt{5}}$$

The volume of the tetrahedron is

$$\frac{1}{3}\text{Area}(\triangle ACD) \cdot PH = \frac{1}{3} \cdot \frac{8 \cdot 2\sqrt{5}}{2} \cdot \frac{2\sqrt{11}}{\sqrt{5}} = \frac{16\sqrt{11}}{3}.$$

11. What is the maximum value of λ such that the following inequality holds for all $a > 0$?

$$a^3 + \frac{1}{a^3} - 2 \geq \lambda(a + \frac{1}{a} - 2)$$

Answer. 9.

Solution. Let $x = f(a) = a + 1/a$. The domain of f is $(0, \infty)$ and its range is $[2, \infty)$, where the minimum of 2 is achieved at $a = 1$; one way to view this fact is through the identity

$$z + \frac{1}{z} = \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^2 + 2, \quad z > 0.$$

In particular, for $a = 1$ the inequality concerned in the problem holds for any λ. Therefore, the solution of the problem is obtained by minimizing the expression

$$\frac{a^3 + \frac{1}{a^3} - 2}{a + \frac{1}{a} - 2} = \frac{\left(a + \frac{1}{a}\right)^3 - 3a \cdot \frac{1}{a}\left(a + \frac{1}{a}\right) - 2}{a + \frac{1}{a} - 2} = \frac{x^3 - 3x - 2}{x - 2},$$

which simplifies to $(x + 1)^2$ via long division, and is minimized at $x = 2$ in the range $[2, \infty)$ of the x-values. Consequently, the minimum of the expression is $(2 + 1)^2 = 9$.

12. Suppose we have

$$\sum_{k=0}^{n-1} \sqrt[3]{\sqrt{ak^3 + bk^2 + ck + 1} - \sqrt{ak^3 + bk^2 + ck}} = \sqrt{n},$$

for all natural numbers n and some constants a, b, c. Find $a - b + c$.

Answer. 1.

Solution. For $k = 0, 1, \ldots$, we should have

$$\sqrt[3]{\sqrt{ak^3 + bk^2 + ck + 1} - \sqrt{ak^3 + bk^2 + ck}} = \sqrt{k+1} - \sqrt{k}.$$

On the other hand,

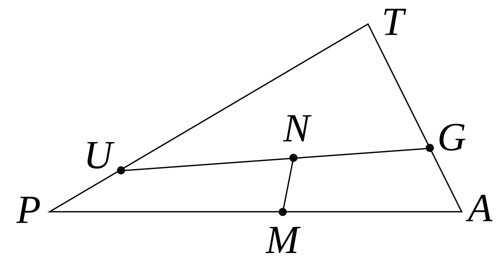
$$\begin{aligned} \left(\sqrt{k+1} - \sqrt{k}\right)^3 &= (k+1)\sqrt{k+1} - 3(k+1)\sqrt{k} + 3k\sqrt{k+1} - k\sqrt{k} \\ &= (4k+1)\sqrt{k+1} - (4k+3)\sqrt{k} \\ &= \sqrt{(4k+1)^2(k+1)} - \sqrt{k(4k+3)^2} \\ &= \sqrt{16k^3 + 24k^2 + 9k + 1} - \sqrt{16k^3 + 24k^2 + 9k}, \end{aligned}$$

hence

$$\sum_{k=0}^{n-1} \sqrt[3]{\sqrt{16k^3 + 24k^2 + 9k + 1} - \sqrt{16k^3 + 24k^2 + 9k}} = \sum_{k=0}^{n-1} \left(\sqrt{k+1} - \sqrt{k}\right) = \sqrt{n}.$$

Taking $a = 16, b = 24, c = 9$ we conclude that $a - b + c = 1$. Notice that such values for a, b, c are unique because the function $\sqrt{x+1} - \sqrt{x}$ is strictly increasing, hence one-to-one, and moreover, any two polynomials that are equal at infinitely many values must be identical (in our case, polynomials of the form $x = ak^3 + bk^2 + ck$ are concerned).

13. In $\triangle PAT$, $\angle P = 36°$, $\angle A = 56°$, and $PA = 10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?



Answer. 80 degrees.

Solution. Place the figure in the coordinate plane with $P = (-5, 0), M = (0, 0), A = (5, 0)$, and T in the first quadrant. Then

$$U = (-5 + \cos 36°, \sin 36°) \quad \text{and} \quad G = (5 - \cos 56°, \sin 56°),$$

and the midpoint N of UG is

$$\left(\frac{1}{2}(\cos 36° - \cos 56°), \frac{1}{2}(\sin 36° + \sin 56°)\right).$$

The tangent of $\angle NMA$ is the slope of the line MN, which is calculated as follows using the sum-to-product trigonometric identities:

$$\begin{aligned} \tan(\angle NMA) &= \frac{\sin 36° + \sin 56°}{\cos 36° - \cos 56°} \\ &= \frac{2 \sin \frac{36°+56°}{2} \cos \frac{36°-56°}{2}}{-2 \sin \frac{36°+56°}{2} \sin \frac{36°-56°}{2}} \\ &= \frac{\cos 10°}{\sin 10°} = \tan 80°. \end{aligned}$$

14. Evaluate the sum

$$\sum_{k=0}^{2017} (-1)^k \cos^{2018}\left(\frac{k\pi}{2018}\right).$$

Answer. $2018/2^{2017}$ or $1009/2^{2016}$.

Solution. Set $\omega = e^{\pi i/2018}$ so that $\omega^{2018} = -1$ and

$$\begin{aligned} S := \sum_{k=0}^{2017} (-1)^k \cos^{2018}\left(\frac{k\pi}{2018}\right) &= \sum_{k=0}^{2017} (-1)^k \left(\frac{\omega^k + \omega^{-k}}{2}\right)^{2018} = \sum_{k=0}^{2017} \omega^{2018k} \left(\frac{\omega^k + \omega^{-k}}{2}\right)^{2018} \\ &= \frac{1}{2^{2018}} \sum_{k=0}^{2017} \left(\omega^{2k} + 1\right)^{2018}. \end{aligned}$$

Using binomial formula we have

$$\begin{aligned} S &= \frac{1}{2^{2018}} \sum_{k=0}^{2017} \sum_{l=0}^{2018} \binom{2018}{l} \omega^{2kl} \\ &= \frac{1}{2^{2018}} \sum_{l=0}^{2018} \binom{2018}{l} \sum_{k=0}^{2017} \left(\omega^{2l}\right)^k. \end{aligned}$$

We note that for $l = 0, 2018$ we have $\left(\omega^{2l}\right)^k = 1$, so $\sum_{k=0}^{2017} \left(\omega^{2l}\right)^k = 2018$, but for $l \neq 0, 2018$

$$\sum_{k=0}^{2017} \left(\omega^{2l}\right)^k = \frac{1-(\omega^{2l})^{2018}}{1-\omega^{2l}} = \frac{1-(\omega^{2018})^{2l}}{1-\omega^{2l}} = 0.$$

Therefore, the sum reduces to two terms only:

$$S = \frac{1}{2^{2018}} \left[\binom{2018}{0} 2018 + \binom{2018}{2018} 2018\right] = \frac{2(2018)}{2^{2018}} = \frac{2018}{2^{2017}}.$$

15. Evaluate

$$\int_0^{\pi/3} \frac{dx}{5+4\cos(2x)}.$$

Answer. $\pi/18$.

Solution. Let $u = \tan x$ so that $dx = du/(1+u^2)$. We also have $\cos(2x) = \cos^2 x - \sin^2 x = (1-u^2)/(1+u^2)$. Substituting these into the integral yields

$$\int_0^{\pi/3} \frac{dx}{5+4\cos(2x)} = \int_0^{\sqrt{3}} \frac{\frac{du}{1+u^2}}{5+4\left(\frac{1-u^2}{1+u^2}\right)} = \int_0^{\sqrt{3}} \frac{du}{5(1+u^2)+4(1-u^2)} = \int_0^{\sqrt{3}} \frac{du}{9+u^2} = \frac{1}{3} \arctan\left(\frac{u}{3}\right)\Big|_0^{\sqrt{3}}$$

$$= \frac{1}{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{18}.$$

16. Evaluate the infinite series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right).$$

Answer. $3\pi/4$.

Solution. We observe that, for $n \geq 2$, we have

$$\frac{2}{n^2} = \frac{\frac{1}{n-1} - \frac{1}{n+1}}{1 + \frac{1}{(n-1)(n+1)}},$$

so that

$$\arctan\left(\frac{2}{n^2}\right) = \arctan\left(\frac{1}{n-1}\right) - \arctan\left(\frac{1}{n+1}\right).$$

Consequently,

$$\begin{aligned}\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right) &= \arctan(2) + \sum_{n=2}^{\infty}\left[\arctan\left(\frac{1}{n-1}\right) - \arctan\left(\frac{1}{n+1}\right)\right] \\ &= \arctan(2) + \lim_{N\to\infty}\left[\arctan(1) + \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{N}\right) - \arctan\left(\frac{1}{N+1}\right)\right] \\ &= \frac{\pi}{4} + \arctan(2) + \arctan\left(\frac{1}{2}\right) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}.\end{aligned}$$

17. Consider the sequence

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Determine $L = \lim_{n\to\infty} x_n$.

Answer. ln(2).

Solution. Note that

$$x_n = \sum_{i=1}^{n} \frac{1}{n+i} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}},$$

which is a (right) Riemann sum for

$$\int_0^1 \frac{dx}{1+x} = \ln(1+x)|_0^1 = \ln(2) - \ln(1) = \ln(2).$$

Therefore, $L = \lim_{n\to\infty} x_n = \ln(2)$.

18. Evaluate the limit

$$\lim_{x\to\infty}\left(\sqrt{x}\cdot\int_x^{x+1} \sin(t^2)\,dt\right).$$

Answer. 0.

Solution. Let

$$I = \int_x^{x+1} \sin(t^2)\,dt = \int_x^{x+1} \frac{2t\sin(t^2)}{2t}\,dt,$$

and integrate by parts to get

$$\begin{aligned} I &= -\left.\frac{\cos(t^2)}{2t}\right|_x^{x+1} - \frac{1}{2}\int_x^{x+1} \frac{\cos(t^2)}{t^2}\,dt \\ &= -\frac{\cos(x+1)^2}{2(x+1)} + \frac{\cos(x^2)}{2x} - \frac{1}{2}\int_x^{x+1}\frac{\cos(t^2)}{t^2}\,dt,\end{aligned}$$

so we have

$$|I| \le \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2}\int_x^{x+1}\frac{1}{t^2}\,dt = \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2}\left.\left(-\frac{1}{t}\right)\right|_x^{x+1} = \frac{1}{x}.$$

Consequently,

$$\left|\sqrt{x} \cdot \int_x^{x+1} \sin(t^2)\, dt\right| \leq \frac{\sqrt{x}}{x},$$

thus, by Squeeze Theorem, we have

$$\lim_{x\to\infty} \left(\sqrt{x} \cdot \int_x^{x+1} \sin(t^2)\, dt\right) = 0.$$

19. Evaluate the following sum.

$$1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \ldots$$

Answer. $-1/e$.

Solution. Our goal is to find the sum

$$S = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^3}{n!}.$$

We have $(n+1)^3 = n(n-1)(n-2) + 6n(n-1) + 7n + 1$, where the coefficients may simply be obtained by substituting $n = 0, 1, 2, 3$. Therefore,

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n(n-1)(n-2) + 6n(n-1) + 7n + 1}{n!} \\
&= \sum_{n=3}^{\infty} \frac{(-1)^n}{(n-3)!} + 6\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} + 7\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} + 6\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - 7\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = -\frac{1}{e},
\end{aligned}$$

using the Maclaurin expansion of e^x.