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The Discrete Universe

Try to solve as much as you can. If you did not succeed to explore the most general case, but were able to treat some particular case, please describe all your findings. In all problems prove your answers.

The goal of this set of problems is to explore the motion of particles in a world that consists of a discrete set of points, for example, points with integer coordinates on a line or a plane. Time in this word takes integer values. For example, assume that our universe consists of all integers on a line. If $x(t)$ is the position of the point at time t in this world, then the discrete velocity $v(t)$ at time $t \geq 1$ is defined by

$$v(t) = x(t) - x(t-1) \tag{1}$$

and the discrete acceleration $a(t)$ at time $t \geq 0$ is defined by

$$a(t) = v(t+1) - v(t). \tag{2}$$

Let $F(x)$ be an integer-valued function of a position which describes the discrete force acting on a particle of mass 1 at a position x. In this case we say the force field F acts on our discrete line and the discrete Newton second law says that if the particle of a unit mass is at the point $x(t)$ at time t, then the discrete acceleration $a(t)$ of this particle at time t satisfies:

$$a(t) = F\big(x(t)\big).$$

and the motion (the trajectory) of the particle is uniquely determined by prescribing the initial position $x(0)$ and the initial velocity $v(0)$.

Further, we say that the trajectory $x(t)$ is *periodic*, if there exists an integer T such that for every integer $t \geq 0$ we have

$$x(t+T) = x(t).$$

The minimal T satisfying this property is called the *period* of the periodic trajectory $x(t)$.

Part 1

Problem 1 Assume that the discrete force field is given by

$$F(x) = -\text{sgn}(x), \text{where } \text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0, \\ -1, & x < 0 \end{cases} \tag{3}$$

Prove that with this force field $F(x)$ any trajectory of the unit mass particle is periodic. Then find its period and amplitude as functions of the initial position and the initial velocity. Here by the *amplitude* A of the periodic trajectory $x(t)$ we mean the half distance between the maximum and minimum value of $x(t)$ for all integer $t \geq 0$.

Solution Given a real number y denote by $\lceil y \rceil$ the ceiling of y, i.e. the least integer number greater than or equal to y. The following function will be important here:

$$s(x) = \left\lceil \frac{-1 + \sqrt{1 + 8|x|}}{2} \right\rceil. \tag{4}$$

The number $s(x)$ has the following meaning: it is the minimal integer t such that $\frac{t(t+1)}{2} \geq |x|$. The expression (4) is obtained by solving the quadratic equation $\frac{t(t+1)}{2} = |x|$ with respect to t and taking the ceiling of its largest root.

Assume that $x(0) = x_0$ and $v(0) = v_0$. First analyze the case of $v_0 = 0$.

Proposition 1. *If $v_0 = 0$, then the corresponding trajectory is periodic with period $T(x_0)$ satisfying*

$$T(x_0) = \begin{cases} 4s(x_0) + 2, & \sqrt{1 + 8|x_0|} \in \mathbb{Z} \\ 4s(x_0), & \sqrt{1 + 8|x_0|} \notin \mathbb{Z} \end{cases} \tag{5}$$

and the amplitude $A(x_0)$ satisfying

$$A(x_0) = \begin{cases} \frac{s(x_0)(s(x_0)+1)}{2}, & \sqrt{1 + 8|x_0|} \in \mathbb{Z} \\ \frac{s(x_0)^2}{2}, & \sqrt{1 + 8|x_0|} \notin \mathbb{Z} \end{cases}. \tag{6}$$

Proof. Without loss of generality we can assume that $x_0 > 0$, because from the fact that the force field is an odd function it follows that the trajectory starting at $-x_0$ at time t is just $-x(t)$. First assume that $1 \le t \le s(x_0)$. Then $x(t-1) > 0$. so $a(t) = F\big(x(t)\big) = -1$. Hence, $v(t) = v(t-1) - 1$. This together with (2) and the assumption that $v(0) = 0$ implies that

$$v(t) = -t. \tag{7}$$

Therefore by (1)

$$x(t) = x_0 - (1 + 2 + \cdots + t) = x_0 - \frac{t(t+1)}{2} \tag{8}$$

Note that the identities (7) and (8) hold for $t = 0$ as well.

Now consider 2 cases separately:

Case 1 Assume that $\sqrt{1 + 8|x_0|}$ is not an integer. Then $x\big(s(x_0)\big)) < 0$. Hence $F\Big(x\big(s(x_0)\big)\Big) = -1$, so

$$v\big(s(x_0) + 1\big) = -(s(x_0) - 1),$$
$$x\big(s(x_0) + 1\big) = x\big(s(x_0)\big) - (s(x_0) - 1) = x_0 - \frac{s(x_0)\big(s(x_0)+1\big)}{2} - (s(x_0) - 1).$$

and similarly for all i such that $0 < i \le s(x_0)$

$$v\big(s(x_0) + i\big) = -(s(x_0) - i),$$
$$x\big(s(x_0) + i\big) = x\big(s(x_0)\big) - (s(x_0) - 1) - \big(s(x_0) - 2\big) - \cdots - \big(s(x_0) - i\big) =$$
$$x_0 - \frac{s(x_0)\big(s(x_0)+1\big)}{2} - \frac{(2s(x_0) - i - 1)i}{2}.$$

In particular, plugging $i = s(x_0)$ into the previous formula, we get

$$v\big(2s(x_0)\big) = 0,$$
$$x\big(2s(x_0)\big) = x_0 - s(x_0)^2.$$

Further note that

$$s\Big(x_0 - s(x_0)^2\Big) = s(x_0),$$

because

$$x\big(2s(x_0)\big) + \big(1 + \cdots + (s(x_0) - 1)\big) = x_0 - s(x_0)^2 + \frac{s(x_0)\big(s(x_0) - 1\big)}{2} =$$
$$x_0 - \frac{s(x_0)\big(s(x_0)+1\big)}{2} < 0,$$

but

$$x\big(2s(x_0)\big) + \big(1 + \cdots + s(x_0)\big) = x_0 - \frac{s(x_0)\big(s(x_0)-1\big)}{2} > 0.$$

Consequently, in the same way as the particle arrives to the point $x_0 - s(x_0)^2$ with velocity 0 after $2s(x_0)$ steps starting from the point x_0 with velocity 0, the particle will return to x_0 after the next $2s(x_0)$ steps and again it has velocity 0. This proves that the trajectory is periodic with the period $4s(x_0)$ and the amplitude $\dfrac{s(x_0)^2}{2}$.

Case 2 Assume that $\sqrt{1+8|x_0|}$ is an integer. Equivalently, $x_0 = \dfrac{s(x_0)\big(s(x_0)+1\big)}{2}$.

The only difference here compared to the previous case is that $x\big(s(x_0)\big)) = 0$. Hence $F\Big(x\big(s(x_0)\big)\Big) = 0$, so

$$\begin{aligned} v\big(s(x_0)+1\big) &= -s(x_0), \\ x\big(s(x_0)+1\big) &= x\big(s(x_0)\big) - s(x_0) = -s(x_0). \end{aligned}$$

Now, proceeding exactly as in the previous step, we get that

$$\begin{aligned} v\big(2s(x_0)+1\big) &= 0, \\ x\big(2s(x_0)+1\big) &= -x_0 = -\frac{s(x_0)\big(s(x_0)+1\big)}{2}. \end{aligned}$$

Since our force field is an odd function we will get that $x\big(2s(x_0)+1+t\big) = -x(t)$ the trajectory for $t = 0, \ldots 2s(x_0)+1$. In particular, $x\big(4s(x_0)+2\big) = -x\big(2s(x_0)+1\big) = x_0$. This proves that $T(x_0) = 4s(x_0)+2$ and also the formula for the amplitude in the considered case. □

Now consider the case of a trajectory $x(t)$ with initial conditions $x(0) = x_0$, $v(0) = v_0$, where v_0 is arbitrary. The idea is to make the reduction to the previous case by showing that there exists time t for which $v(t) = 0$. Here we use that if $x(t)$ is a trajectory of our discrete system with velocity $v(t)$ at time t, then for any integer C the function $x(t) := x(t-C)$ is a trajectory as well, with velocity $v(t-C)$. Moreover, if $x(t)$ is periodic then also $\tilde{x}(t)$ is periodic with the same period and amplitude.

Note that in some cases it is more convenient to show existence of a negative t such that $v(t) = 0$ and it will be also enough for our purposes.

Consider several cases separately:

1 *If* $x_0 > 0$ *and* $v_0 > 0$, then similarly to above it can be shown that $v(v_0) = 0$. Then

$$x(v_0) = x_0 + \frac{v_0(v_0 - 1)}{2}. \tag{9}$$

Since $\tilde{x}(t) = x(t + v_0)$ has zero velocity at 0, it is periodic by Proposition 1. Hence, the trajectory $x(t)$ is periodic with the same period and the same amplitude as $\tilde{x}(t)$, i.e. if $T(x_0, v_0)$ and $A(x_0, v_0)$ are the period and the amplitude of the trajectory $x(t)$, then

$$T(x_0, v_0) = T\left(x_0 + \frac{v_0(v_0 - 1)}{2}\right) \tag{10}$$

$$A(x_0, v_0) = A\left(x_0 + \frac{v_0(v_0 - 1)}{2}\right), \tag{11}$$

where the functions $T(x)$ and $A(x)$ are as in (5) and (6), respectively.

2. *If* $x_0 \geq 0$ *and* $v_0 < 0$, then it is more convenient to go backward in time to obtain that $v(v_0) = 0$ and that $x(v_0)$ satisfies formula (9), so that $x(t)$ is periodic with the period $T(x_0, v_0)$ and the amplitude $A(x_0, v_0)$ satisfying formulas (10) and (11), respectively.

3. The case of $x_0 < 0$ can be reduced to the case $x_0 > 0$ because if $x(t)$ is a trajectory of our system with velocity $v(t)$ then $-x(t)$ is also a trajectory of our system as well with velocity $-v(t)$. So, the trajectory is periodic with the periods $T(x_0, v_0) = T(-x_0, -v_0)$ and the amplitude $A(x_0, v_0) = A(-x_0, -v_0)$, where the right-hand sides are computed by formulas (10) and (11), i.e.

$$T(x_0, v_0) = T(-x_0, -v_0) = T\left(-x_0 + \frac{v_0(v_0 + 1)}{2}\right) \tag{12}$$

$$A(x_0, v_0) = A(-x_0, -v_0) = A\left(-x_0 + \frac{v_0(v_0 + 1)}{2}\right), \tag{13}$$

Formulas (10)-(11) and (12)-(13) can be unified as follows:

$$T(x_0, v_0) = T\left(|x_0| + \frac{v_0(v_0 - \theta(x_0))}{2}\right) \tag{14}$$

$$A(x_0, v_0) = A\left(|x_0| + \frac{v_0(v_0 - \theta(x_0))}{2}\right), \tag{15}$$

where

$$\theta(x) = \begin{cases} 1, & x > 0 \\ -1, & x \le 0 \end{cases}. \tag{16}$$

Problem 2 The *conservation law* for a given force field $F(x)$ is any *nonconstant* function $I(x, v)$ such that

$$I\big(x(t), v(t)\big) = I\big(x(0), v(0)\big), \qquad \text{for every } t \ge 0,$$

where $x(t)$ is any trajectory of the unit mass particle in the force field F and $v(t)$ is its velocity.

Find at least one conservation law for the trajectories of unit mass particle in the force field of problem 1.

Solution This problem obviously does not have a unique answer. The period (or amplitude) are clearly constant on every trajectory and they are not constant functions. The analytic expression for $T(x, v)$ (resp. $A(x, v)$) are given by formulas (14) and (5) ((15) and (6) respectively). Another possible conservation law is the maximal position on each trajectory, analytic expressions for which as a function of x_0 and v_0 can be deduced from the arguments on the previous page.

Remark 1. *Note also that if I is a conservation law then for any function $f : \mathbb{N} \to \mathbb{R}$ the function $f\big(I(x, v)\big)$ (if it is not constant) defines another conservation law. Moreover, if I is a conservation law such that it takes different values on different trajectories (for example f I is equal to the maximal position on a trajectory), then any other conservation law is of the form $f\big(I(x, v)\big)$ for some function $f : \mathbb{N} \to \mathbb{R}$.*

Problem 3 Study problems 1 and 2 under the assumption that the discrete universe is not $\mathbb{Z}$ but $\mathbb{Z} + \frac{1}{2}$, i.e. the set of all points with fractional part $1/2$ (note that in this case still $v(t) \in \mathbb{Z}$). Here it is not necessary to give as detailed solution as in the problems 1 and 2. Just indicate the similarities and differences in the results and conclusions in this setting compared to the previous one.

Solution. The main difference in the present setting is that the particle never hits the origin making the treatment much easier compared to the previous one. The function $s(x)$ is the same as in (4) and the analytic expressions for $T(x_0)$, $T(x_0, v_0)$, $A(x_0)$, $A(x_0, v_0)$ are the same as in (5), (14), (6), and (15), respectively, for the case of $\sqrt{1 + 8|x_0|} \notin \mathbb{Z}$, which is always the case here. The conservation law can be taken as $T(x, v)$ and nontrivial composition of any function with it.

Problem 4 Now consider the motion of two unit mass particles in the universe consisting of $\mathbb{Z}$ such that if $x_1(t)$ and $x_2(t)$ are positions at time t of the first and the second particle,

respectively, then the discrete acceleration $a_1(t)$ at time t of the first particle is equal to $\operatorname{sign}(x_2(t) - x_1(t))$ and the discrete acceleration $a_2(t)$ at time t of the second particle is equal to $\operatorname{sign}\big(x_1(t) - x_2(t)\big)$. Study problems 1 and 2 for this setting, namely

(a) Find all initial positions and velocities of particles for which the corresponding trajectories $\big(x_1(t), x_2(t)\big)$ are periodic, i.e. there exist $T > 0$ such that

$$x_1(t+T) = x_1(t), \quad x_2(t+T)\big) = x_2(t), \quad \forall t \geq 0.$$

For every periodic solutions find its period.

(b) Find two independent conservation laws for the given force field. Independent means that one conservation law is not a composition of a single variable function with the other conservation law.

(c) Describe all (not only periodic) trajectories of the system.

Solution 4 (a). Let

$$c(t) = \frac{1}{2}\big(x_1(t) + x_2(t)\big) \tag{17}$$

$$d(t) = \frac{1}{2}\big(x_1(t) - x_2(t)\big) \tag{18}$$

Remark 2. *Note that $c(t)$ represent the motion of the center of mass of the particle and $d(t)$ the evolution of the signed distance between them.*

Both $c(t)$ and $d(t)$ take values in

$$\frac{1}{2}\mathbb{Z} = \mathbb{Z} \cup \left(\mathbb{Z} + \frac{1}{2}\right)$$

Obviously the trajectory $c(t)$ moves with zero acceleration and therefore with the constant velocity equal to $v_1(0) + v_2(0)$. Therefore if the trajectory $(x_1(t), x_2(t))$ is periodic, then its center of mass does not move, i.e.

$$v_2(0) = -v_1(0). \tag{19}$$

Now let us prove that the condition (19) is sufficient for the periodicity of the trajectory. Clearly the trajectory $(x_1(t), x_2(t))$ is periodic if and only if the trajectory

$(c(t), d(t))$, related to $(x_1(t), x_2(t))$ by (17) and(18) is periodic. Since $c(t)$ is constant by assumption in (19), it is enough to prove that $d(t)$ is periodic. Note that $d(t)$ has acceleration exactly equal to the force $F(d)$, as defined in (3). Therefore by Remark 2 and the conclusions of problems 1 and 3 $d(t)$ is periodic. Moreover, since $c(t)$ is constant, the period of our original trajectory $(x_1(t), x_2(t))$ is equal to the period of $d(t)$, which in turn is equal to

$$T\left(\frac{x_1(0)-x_2(0)}{2}, \frac{v_1(0)-v_2(0)}{2}\right) \overset{(19)}{=} T\left(\frac{x_1(0)-x_2(0)}{2}, v_1(0)\right),$$

where T is defines by (14).

Solution 4 (b). Since the velocity of $c(t)$ is constant, it defines the conservation law

$$I_1(x_1, x_2, v_1, v_2) := \frac{v_1+v_2}{2}. \tag{20}$$

Since $d(t)$ is always periodic (also when condition (19) does not hold) its period defines another conservation law

$$I_2(x_1, x_2, v_1, v_2) := T\left(\frac{x_1-x_2}{2}, \frac{v_1-v_2}{2}\right), \tag{21}$$

where the function $T(x, v)$ is defined by (14).

Obviously these two functions are independent, because the period of $d(t)$ is independent of the initial velocity of the center of mass.

Solution 4 (c). The general trajectory $\big(x_1(t), x_2(t)\big)$ is such that its center of mass moves with constant velocity, while in the frame of the center of mass (i.e. when we subtract the coordinates of the center of mass from the coordinate of each particle) both particles moves periodically and their positions have opposite sign at every time moment.

Part 2

In this part our universe consists of the vertices of a regular n-gon $P_0P_1 \dots P_{n-1}$. Let O be the center of the n-gon. The particle moves in this universe by jumping from one vertex to another vertex of the n-gon. Let $x(t)$ be the position of the particle at time t. Assume that $x(0) = P_0$ and for every time $t > 0$ the angle between the ray

$Ox(t-1)$ and $Ox(t)$, counted in the counterclockwise direction, is equal to $\frac{2\pi}{n}t$. In other words, the particle moves in our universe with constant angular acceleration 1, starting at P_0 with 0 angular velocity. Further, let $A(n)$ denote the number of different vertices of the n-gon that are visited by a particle during its motion. The goal of the following exercises is to express $A(n)$ in terms of the prime factorization of n.

Problem 5 Prove that for every n the trajectory $x(t)$ is periodic and find its period $T(n)$.

Answer

$$T(n) = \begin{cases} n, & n \text{ is odd} \\ 2n, & n \text{ is even} \end{cases}. \tag{22}$$

Solution Given t let $r(t)$ be the remainder of the division of $\frac{t(t+1)}{2}$ by n. Then clearly $x(t) = P_{r(t)}$. Note that $r(2n) = 0$, because $\frac{2n(2n+1)}{2} = n(2n+1)$ and the angle between $Ox(2n+t-1)$ and $Ox(2n+t)$ is equal to

$$\frac{2\pi}{n}(2n+t) = 4\pi + \frac{2\pi}{n}t \equiv \frac{2\pi}{n}t \mod 2\pi.$$

This implies that

$$x(t+2n) = x(t), \tag{23}$$

i.e., $x(t)$ is periodic.

Let us prove that the period $T(n)$ satisfies (22). Indeed, if T is a multiple of the period, then

$$r(T+1) = 1. \tag{24}$$

The latter implies that $T + 1 \equiv 1 \mod n$, i.e. that T must be divisible by n. In particular $T(n)$ is divisible by n, while (23) implies that $T(n)$ divides $2n$. Therefore $T(n)$ is either n or $2n$.

Further, if T is a multiple of a period, then

$$r(T) = 0 \Leftrightarrow \frac{T(T+1)}{2} \equiv 0 \mod n \tag{25}$$

Moreover, if T satisfies both (24) and (25) then T is a multiple of a period. Note that if n is even, then $t = n$ does not satisfy (25) so in this case $T(n) = 2n$, while if n is odd then $T = n$ does satisfy both (24) and (25), so in this case $T(n) = n$. This completes the proof of (22). □

Problem 6 Find $A(n)$ if

(a) $n = 2^k$.

(b) n is an odd prime.

Prove your answers.

Answers: $A(2^k) = 2^k$; if $n = p$ is an odd prime, then

$$A(n) = \frac{n+1}{2}. \tag{26}$$

Proof. As before, given t let $r(t)$ be the remainder of the division of $\dfrac{t(t+1)}{2}$ by n. First prove the following lemma

Lemma 1. *For every n and $i = 0, \dots, T(n) - 1$*

$$r(i) = r\big(T(n) - 1 - i\big) \tag{27}$$

Proof. Consider the case of even and odd n separately:

Case 1: *n is even.* Then by the previous problem $T(n) = 2n$ and we have

$$r(2n-1-i)-r(i) \equiv \sum_{k=i+1}^{2n-1-i} k \bmod n \equiv \sum_{k=i+1}^{2n-1-i} (k-n) \bmod n \equiv \sum_{l=-n+i+1}^{n-i-1} l \bmod n \equiv 0 \bmod n,$$

i.e. $r(2n-1-i) - r(i) = 0$, q.e.d.

Case 2: *n is odd.* Then by the previous problem $T(n) = n$ and we have

$$r(n-1-i) - r(i) \equiv \sum_{k=i+1}^{n-1-i} k \bmod n \equiv \left(\sum_{k=i+1}^{\frac{n-1}{2}} k + \sum_{\frac{n+1}{2}}^{n-1-i} k \right) \bmod n \equiv$$

$$\left(\sum_{k=i+1}^{\frac{n-1}{2}} (k-n) + \sum_{\frac{n+1}{2}}^{n-1-i} k \right) \bmod n \equiv \sum_{k=-n+i+1}^{-\frac{n+1}{2}} k + \sum_{\frac{n+1}{2}}^{n-1-i} k \bmod n \equiv 0 \bmod n,$$

i.e. $r(n-1-i) - r(i) = 0$, q.e.d. □

The lemma immediately implies that

Corollary 1. *All possible values of* $r(i)$ *appear for* $i = 0, \ldots, \left\lceil \frac{T(n)}{2} \right\rceil - 1$, *i.e. for* $i = 0, \ldots, n-1$ *if* n *is even and for* $i = 0, \ldots, \frac{n-1}{2}$ *if* n *is odd.*

□

Consequently, to prove our answers for both parts of the problem it is enough to show that for $n = 2^k$ or n being an odd prime $r(i_1) \neq r(i_2)$ for all distinct i_1 and i_2 from the set $\{0, \ldots, \left\lceil \frac{T(n)}{2} \right\rceil - 1\}$.

Assume WLOG that $i_2 < i_1$. Then

$$r(i_1) - r(i_2) \equiv \sum_{k=i_2+1}^{i_1} k \bmod n = \frac{(i_1 + i_2 + 1)(i_1 - i_2)}{2} \bmod n$$

Assume by contradiction that

$$\frac{(i_1 + i_2 + 1)(i_1 - i_2)}{2} \equiv 0 \bmod n \tag{28}$$

or, equivalently $(i_1 + i_2 + 1)(i_1 - i_2)$ is divisible by $2n$. Now consider the cases of items (a) and (b) separately:

The case of item (a) $n = 2^k$. Since the factors $i_1 + i_2 + 1$ and $i_2 - i_1$ have to be of the different parity, the one of them that is even must be divisible by $2n = 2^{k+1}$ but both of these factors are smaller than $2n$ by Corollary 1 and are not equal to zero. So, we get a contradiction , and $r(i_1) \neq r(i_2)$ for all distinct i_1 and i_2 from the set $\{0, \ldots, n-1\}$, which gives that $A(n) = n$.

The case of item (b) n is odd prime. Since $(i_1 + i_2 + 1)(i_1 - i_2)$ is divisible by $2n$ and n is prime, either $i_1 + i_2 + 1$ or $i_2 - i_1$ is divisible by n, but from Corollary 1 both of this factors are smaller than n and are not equal to zero. So, we get a contradiction, and $r(i_1) \neq r(i_2)$ for all distinct i_1 and i_2 from the set $\{0, \ldots, \frac{n-1}{2}\}$, which gives that $A(n) = \frac{n+1}{2}$.

Problem 7 (a) Find a relation between $A(2n)$ and $A(n)$. Prove your answer.

(b) Let m and n be odd and coprime. Find a relation between $A(m)$, $A(n)$, and $A(mn)$. Prove your answer.

Answers: **(a)** $A(2n) = 2A(n)$; **(b)** $A(mn) = A(m)A(n)$.

Proof of 7(a) Given n and t let $r_n(t)$ be the remainder of the division of $\dfrac{t(t+1)}{2}$ by n and let Δ_n be the set of all such remainders for a given n. Also given a subset X of integers and an integer k denote by $X + k$ the subset consisting of all elements obtained by adding k to an element of x. First, clearly

$$\Delta_{2n} \subseteq \Delta_n \cup (\Delta_n + n).$$

Moreover, for every t at least one of the elements $r(t)$ or $r(t) + n$ belongs to Δ_n. Let us prove that for any t both of this elements must belong to Δ_{2n}.

Assume by contradiction that for some $t = i$ only one of these two numbers appear in Δ_n . From Lemma 1 and the definition of the period

$$r_n(i) = r_n\big(T(n) - 1 - i\big) = r_n\big(i + T(n)\big) = r_n\big(2T(n) - 1 - i\big).$$

Then by our assumption

$$r_{2n}(i) = r_{2n}\big(T(n) - 1 - i\big) = r_{2n}\big(i + T(n)\big) = r_{2n}\big(2T(n) - 1 - i\big) \tag{29}$$

By the same arguments as the ones which lead to (28) we get that if $\{i_2, i_1\} \subset \{i, T(n) - 1 - i, i + T(n), 2T(n) - 1 - i\}$, then $(i_1 + i_2 + 1)(i_2 - i_1)$ must be divisible by $2 \cdot 2n = 4n$. In particular, if we take $(i_1, i_2) = \big(i, , T(n) - 1 - i\big)$ then

$$T(n)\big(T(n) - 1 - 2i\big) \quad \text{is} \quad \text{divisible by } 4n, \tag{30}$$

and if we take $(i_1, i_2) = \big(i, i + T(n)\big)$,then

$$T(n)\big(T(n) + 2i + 1\big) \quad \text{is} \quad \text{divisible by } 4n. \tag{31}$$

Now consider the cases of even and odd n separately:

- If n is even, then by (22) $T(n) = 2n$ and plugging it to (30) we get that $2n - 1 - 2i$, which is always odd, is divisible by 2, which is absurd.
- If n is odd, then by (22) $T(n) = n$. If we plug this into (30) we get that $n - 1 - 2i$ is divisible by 4. On the other hand, if we plug it into (31), then we get that $n + 2i + 1$ is divisible by 4. Subtracting the numbers in these two conclusions we get that $4i + 2$ is divisible by 4, which is absurd.

□

Proof of 7(b)

The number $a \in \{0, \dots n-1\}$ is called a *quadratic residue* $\bmod n$, if there exists x such that $x^2 = a \bmod n$. First prove the following lemma that will be also used in the solution of Problem 8.

Lemma 2. *If n is odd, then $A(n)$ is equal to the number $Q(n)$ of quadratic residues modulo n.*

Proof. This lemma is based on the fact that if $m = r(i)$ for some i. then

$$8m + 1 \equiv 4i^2 + 4i + 1 \bmod n \equiv (2i + 1)^2 \bmod n,$$

i.e. $8m + 1$ is the a quadratic residue. The correspondence between the set of all $r(i)$ and all quadratic residues $\bmod n$ is bijective for odd n, because the map $m \mapsto 8m + 1 \bmod n$ from the set $\mathbb{Z}_n$ of all remainders mod n to itself is bijective in this case. □

By Lemma 2 our answer will follow from the following identity

$$Q(mn) = Q(m)Q(n), \quad \forall \text{ odd } m, n. \tag{32}$$

where $Q(n)$ is the number of the quadratic residue $\bmod n$.

To prove (32) we use the Chinese Remainders theorem, which states that if m and n are coprime, then for every given α and β the system of congruences

$$x \equiv \alpha \bmod m, \tag{33}$$

$$x \equiv \beta \bmod n \tag{34}$$

has a unique solution $x = \phi(\alpha, \beta) \bmod mn$. The map $\phi : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_{mn}$ preserves the operation of modular multiplication, i.e.

$$\phi(\alpha_1\alpha_2 \bmod m, \beta_1\beta_2 \bmod n) = \phi(\alpha_1, \beta_1)\phi(\alpha_2, \beta_2) \bmod mn. \tag{35}$$

Now we are ready to prove (2): If a is a quadratic residue $\bmod mn$, then there exists b such that $b^2 = a \bmod mn$. Let $(x, y) = \phi^{-1}(b)$, and $(\alpha, \beta) = \phi^{-1}(a)$. Then by property (35)

$$x^2 \equiv \alpha \bmod m, \quad y^2 \equiv \beta \bmod n,$$

i.e., to any quadratic residue $\bmod mn$ we uniquely assigned an ordered pair consisting of a quadratic residual $\bmod m$ and a quadratic residue $\bmod n$. This implies that $Q(mn) \le Q(m)Q(n)$.

In the opposite direction, if x and y are quadratic residues $\bmod m$ and a quadratic residue $\bmod n$, respectively, then there exist z and w such that $z^2 \equiv x \bmod m$ and $w^2 \equiv y \bmod n$. Then, using ϕ we obtain that $\phi(x,y) \equiv \phi(z,w)^2 \bmod mn$, i.e. $\phi(x,y)$ is a quadratic residue $\bmod mn$. The bijectivity of ϕ implies that $Q(mn) \geq Q(m)Q(n)$. Thus, combining both inequalities, we get (32) and therefore finish the proof of the item.

Problem 8 Given n denote by Δ_n the set of all possible remainders obtained after the division of numbers of the form $\dfrac{t(t+1)}{2}$ by n (so that $A(n)$ is the number of elements of Δ_n). In the present problem you can use the following fact without proof: for $n = p^k$, where p is an odd prime, the number of elements m in Δ_n such that the number $8m+1$ is not divisible by p is equal to $\dfrac{p^k - p^{k-1}}{2}$. Based on this fact, find $A(n)$ if

(a) $n = p^2$, where p is an odd prime.

(b) $n = p^k$, where $k \geq 3$ and p is odd prime.

Answers:

8(a)

$$A(p^2) = \frac{p^2 - p + 2}{2} \tag{36}$$

8(b) For $k \geq 3$

$$A(p^k) = \begin{cases} \dfrac{p^{k+1} + p + 2}{2(p+1)} & k \text{ is even}, \\ \dfrac{p^{k+1} + 2p + 1}{2(p+1)} & k \text{ is odd}. \end{cases} \tag{37}$$

Remark 3. *Note that the result of both items can be uniformly written as*

$$A(p^k) = \lceil \frac{p^{k+1}}{2(p+1)} \rceil$$

Proof. By Lemma 2 $A(n) = Q(n)$ where $Q(n)$ is the number of quadratic residues mod n. Since, as mentioned in the proof of Lemma 2, $m \in \Delta_n$ with odd n if and only if $8m+1$ is a quadratic residue mod n and also the map $m \mapsto 8m+1$ is bijective on the set of remainders mod n, the fact given in the formulation of the problem is equivalent to the following one: for $n = p^k$, where p is an odd prime number, the

number $C(n)$ of quadratic residues, which are also coprime to n (equivalently, not divisible by p in the considered case) satisfies

$$C(p^k) = \frac{p^k - p^{k-1}}{2}. \tag{38}$$

Proof of 8(a) If a is a quadratic residue mod p^2 which divides p, then it must divide p^2, i.e. it must be equal to 0. Therefore

$$A(p^2) = Q(p^2) = C(p^2) + 1 = \frac{p^2 - p}{2} + 1 = \frac{p^2 - p + 2}{2}.$$

Proof of 8(b) The proof is based on the following

Lemma 3. *m is a quadratic residue* mod p^k *which is divisible by p if and only if $m \equiv lp^2 \bmod p^k$, where l is a quadratic residue* mod p^{k-2}*. Moreover, such l is unique* mod p^{k-2}*.*

Proof. Let l be a quadratic residue mod p^{k-2} then there exist x such that $l \equiv x^2 \bmod p^{k-2}$. Hence $lp^2 \equiv (xp)^2 \bmod p^k$, i.e. lp^2 is a quadratic residue mod p^k.

In the opposite direction, if m is a quadratic residue mod p^k which is divisible by p, then there is z such that $m \equiv z^2 \bmod p^k$ and z^2 is divisible by p. The latter implies that z is divisible by p, i.e. there exists y such that $z = yp$. Therefore $m \equiv z^2 \equiv y^2p^2 \bmod p^k$, i.e. $m = y^2p^2 + cp^k = (y^2 + cp^{k-2})p^2$ for some c. Then $l = y^2 + cp^{k-2}$ is the desired quadratic residue mod p^{k-2}. The uniqueness of such l mod p^{k-2} is obvious. □

As a consequence of the previous Lemma, we get that the number of l is a quadratic residues mod p^k that are divisible by p is equal to the number of all quadratic residues mod p^{k-2}. Hence we have the following recursive formula

$$A(p^k) = A(p^{k-2}) + C(p^k), \quad k \geq 3. \tag{39}$$

Using this formula inductively we obtain that

- If $k \geq 3$ is odd, then

$$A(p^k) = C(p^k) + C(p^{k-2}) + \cdots C(p^3) + A(p)$$

Using (38),the result of Problem 6 (b), and the formula for the sum of geometric progression, we get

$$A(p^k) = \frac{p^k - p^{k-1}}{2} + \frac{p^{k-2} - p^{k-3}}{2} + \cdots + \frac{p^3 - p^2}{2} + \frac{p+1}{2}$$
$$= \frac{(-p^2)(1-(-p)^{k-1})}{2(1-(-p))} + \frac{p+1}{2} = \frac{p^{k+1} - \cancel{p^2} + \cancel{p^2} + 2p + 1}{2(p+1)} = \frac{p^{k+1} + 2p + 1}{2(p+1)}.$$

- If $k \geq 4$ is even, then

$$A(p^k) = C(p^k) + C(p^{k-2}) + \cdots + C(p^4) + A(p^2)$$

Using (38),the result of item (a) of this problem, and the formula for the sum of geometric progression, we get

$$A(p^k) = \frac{p^k - p^{k-1}}{2} + \frac{p^{k-2} - p^{k-3}}{2} + \cdots + \frac{p^4 - p^3}{2} + \frac{p^2 - p + 2}{2}$$
$$= \frac{(-p^3)(1-(-p)^{k-2})}{2(1-(-p))} + \frac{p^2 - p + 2}{2} = \frac{p^{k+1} - \cancel{p^3} + \cancel{p^3} - \cancel{p^2} + 2p + \cancel{p^2} - p + 2}{2(p+1)} =$$
$$\frac{p^{k+1} + p + 2}{2(p+1)}.$$

□

Problem 9 Express $A(n)$ in terms of the prime factorization of n for arbitrary n.

Solution From Problem 7 and 6(a) if $n = 2^{k_0} p_1^{k_1} \cdots p_l^{k_l}$ is the prime factor decomposition of n, then

$$A(n) = 2^{k_0} A\big(p_1^{k_1}\big) \cdots A\big(p_l^{k_l}\big),$$

where (by Problems 6(b) and 8) $A\big(p_i^{k_i}\big)$ satisfies (26), (36), or (37) depending on k_i.

Problem 10 Find the maximal and the minimal values among all ℓ with the following property: there exists a sequence of natural indices $\{n_k\}_{k=1}^{\infty}$ with $n_k < n_{k+1}$ such that $\ell = \lim_{k\to\infty} \dfrac{A(n_k)}{n_k}$.

Answer: the maximal value is 1 and the minimal value is 0.

Solution. A number ℓ for which there exists a sequence of natural indices $\{n_k\}_{k=1}^{\infty}$ with $n_k < n_{k+1}$ such that $\ell = \lim\limits_{k\to\infty} \dfrac{A(n_k)}{n_k}$ is called a *partial limit* of the sequence $\dfrac{A(n)}{n}$.

Obviously $0 \le A(n) \le n$ or, equivalently $0 \ge \dfrac{A(n)}{n} \le 1$. Therefore, if ℓ is a partial limit, then $0 \le \ell \le 1$. On the other hand, by Problem 6(a) $A(2^k) = 2^k$, so that $\dfrac{A(2^k)}{2^k} = 1$. Hence, the maximal value of ℓ is 1.

Let us prove that the minimal value is 0. For this first prove that for every natural m the number $\dfrac{1}{2^m}$ is a partial limit of the considered sequence. Indeed, enumerate all odd prime numbers in the increasing order by $p_1, p_2, p_3, \ldots$ (so that $p_1 = 3$, $p_2 = 5$, etc). Let

$$n_{m,k} = \prod_{i=k}^{k+m-1} p_i$$

Then by the result of Problem 6(b) and Problem 7 (b)

$$A(n_{m,k}) = \frac{1}{2^m} \prod_{i=k}^{k+m-1} (p_i + 1)$$

so that

$$\frac{A(n_{m,k})}{n_{m,k}} = \frac{1}{2^m} \prod_{i=k}^{k+m-1} \frac{p_i + 1}{p_i} \ge \frac{1}{2^m}$$

Hence, since $p_i \to \infty$ as $i \to \infty$, we have

$$\lim_{k\to\infty} \frac{A(n_{m,k})}{n_{m,k}} = \frac{1}{2^m},$$

so that $\dfrac{1}{2^m}$ is indeed a partial limit.

Now, for every m we can find $k(m)$ such that

$$\frac{1}{2^m} \le \frac{A(n_{m,k(m)})}{n_{m,k(m)}} \le \frac{1}{2^m} + \frac{1}{m}$$

Sending m to ∞ we get by the Sandwich Theorem that

$$\lim_{m\to\infty} \frac{A(n_{m,k(m)})}{n_{m,k(m)}} = 0,$$

q.e.d.