BEST STUDENT EXAM
Texas A&M High School Math Contest
November 9, 2019

Directions: Answers should be simplified, and if units are involved include them in your answer.

1. Solve the equation $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$.

   Answer. 3/2

   Solution. The equation can be written as
$$4^x + 4^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}},$$
   or in other words,
$$4^{x-\frac{1}{2}}\left(4^{\frac{1}{2}} + 1\right) = 3^{x-\frac{1}{2}}(3+1).$$
   Upon regrouping the terms by division, we get
$$\left(\frac{3}{4}\right)^{x-\frac{1}{2}} = \frac{3}{4},$$
   hence $x - 1/2 = 1$ and $x = 3/2$.

2. A set containing precisely two elements is called a *doubleton*. In how many different ways can we choose three doubeton subsets of $\{1,2,3,4,5,6\}$ in such a way that every pair of them has exactly one common element?

   Answer. 80

   Solution. There are two cases:

   (1) One of the elements in the union of the three doubletons appears in the three of them, which is possible in $6 \times \binom{5}{3} = 60$ ways.

   (2) Each element in the union of the three doubletons appears in exactly two of them, which is possible in $\binom{6}{3} = 20$ ways.

   So there is a total of 80 ways.

3. Chris thinks of three different prime numbers, and notices that their product is equal to 19 times their sum. Compute the sum of the three prime numbers Chris must be thinking of.

   Answer. 33

   Solution. We notice right away that one of Chris's three primes must be 19. We call the other two p and q, and notice thus that
$$19pq = 19(19 + p + q)$$

$$pq = 19 + p + q$$

$$pq - p - q = 19$$

$$(p-1)(q-1) = 20$$

due to Simon's Favorite Factoring Trick. Thus we need two integers whose product is 20 that are both one less than a prime. We see that $4 \cdot 5$ and $1 \cdot 20$ both do not work, but $2 \cdot 10$ does, so we have 3 and 11 as Chris's other two primes. So the three primes must be $3, 11, 19$ with the sum of 33.

4. Let

$$x = \frac{18}{(1+\sqrt{19})(1+\sqrt[4]{19})(1+\sqrt[8]{19})(1+\sqrt[16]{19})(1+\sqrt[32]{19})(1+\sqrt[64]{19})}.$$

Find $(1+x)^{128}$.

Answer. 361

Solution. If we multiply and divide x by $\sqrt[64]{19} - 1$, we get

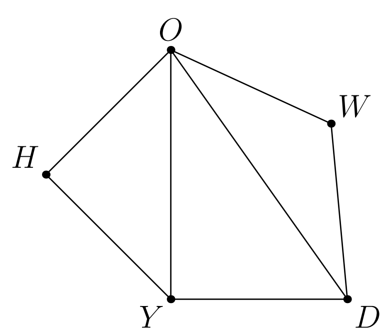
$$x = \frac{18(\sqrt[64]{19}-1)}{(1+\sqrt{19})(1+\sqrt[4]{19})(1+\sqrt[8]{19})(1+\sqrt[16]{19})(1+\sqrt[32]{19})\underbrace{(1+\sqrt[64]{19})(\sqrt[64]{19}-1)}_{\sqrt[32]{19}-1}}.$$

The denominator is, now, simplified to 18, so $x = \sqrt[64]{19} - 1$ and $(1+x)^{128} = 19^2 = 361$.

5. In equilateral pentagon $HOWDY$, $\angle H = 90°$ and $\angle W = 120°$. Compute $\angle Y$ (in degrees).

Answer. 135 degrees

Solution. Without loss of generality, let the sides of $HOWDY$ have length 1. Then we see that since $HO = HY = 1$ and $\angle H = 90°$, we must have $OY = \sqrt{2}$ by the Pythagorean Theorem. Similarly, since $WO = WD = 1$ and $\angle W = 120°$, we must have $OD = \sqrt{3}$. (This can be shown either by the Law of Cosines, or more simply by dropping the altitude from W to $\overline{OD}$, which creates two $30-60-90$ triangles.) Finally, $DY = 1$. But then ODY is a right triangle with right angle at Y, since $DY^2 + OY^2 = OD^2$. Thus, $\angle OYD = 90°$, and $\angle HYD = \angle HYO + \angle OYD = 45° + 90° = 135°$.



6. Evaluate the following limit.

$$\lim_{x\to 1}\left(\frac{3}{1-\sqrt{x}}-\frac{2}{1-\sqrt[3]{x}}\right)$$

Answer. 1/2

Solution. Let L denote the desired limit, and set $x=y^6$, so we have

$$L=\lim_{y\to 1}\left(\frac{3}{1-y^3}-\frac{2}{1-y^2}\right)=\lim_{y\to 1}\left(\frac{3}{(1-y)(1+y+y^2)}-\frac{2}{(1-y)(1+y)}\right)$$
$$=\lim_{y\to 1}\frac{1+y-2y^2}{(1-y)(1+y)(1+y+y^2)}=\lim_{y\to 1}\frac{1+2y}{(1+y)(1+y+y^2)}=\frac{1}{2}.$$

7. Let $N=10004000600040001$. Find $\lfloor\sqrt[4]{N}\rfloor$, where $\lfloor x\rfloor$ denotes the largest integer that is not greater than x.
Answer. 10001

Solution. Notice that

$$N=1\times 10^{16}+4\times 10^{12}+6\times 10^{8}+4\times 10^{4}+1,$$

which is to be compared to the binomial expansion

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

with $a=10^4$ and $b=1$. Therefore, $N=(10^4+1)^4$.

8. Simplify the expression $S=\sqrt[3]{9+4\sqrt{5}}+\sqrt[3]{9-4\sqrt{5}}$.

Answer. 3

Solution. Let $a=\sqrt[3]{9+4\sqrt{5}}$ and $b=\sqrt[3]{9+4\sqrt{5}}$. Then we have

$$S^3=(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b).$$

We immediately have $a^3+b^3=18$ and $ab=\sqrt[3]{81-16\times 5}=1$, hence $S^3=18+3S$. We have the factorization

$$S^3-3S-18=(S-3)(S^2+3S+6),$$

and the quadratic part is irreducible, so $S=3$.

9. Let

$$P_n=\frac{7}{9}\times\frac{18}{20}\times\frac{33}{35}\times\cdots\times\frac{2n^2+n-3}{2n^2+n-1},$$

for $n\geq 2$. Find $\lim_{n\to\infty}P_n$.

Answer. 8/15

Solution. Notice that

$$\frac{2n^2+n-3}{2n^2+n-1} = \frac{(2n+3)(n-1)}{(2n-1)(n+1)}.$$

Since $n+1=(n+2)-1$ and $2n+3=2(n+2)-1$, all but eight factors in the product P_n cancel each other out for large enough n. More precisely, we have

$$P_n = \frac{(7)(1)}{(3)(3)} \times \frac{(9)(2)}{(5)(4)} \times \frac{(11)(3)}{(7)(5)} \times \frac{(13)(4)}{(9)(6)} \times \cdots \times \frac{(2n-1)(n-3)}{(2n-5)(n-1)} \times \frac{(2n+1)(n-2)}{(2n-3)(n)} \times \frac{(2n+3)(n-1)}{(2n-1)(n+1)}$$

so

$$P_n = \frac{(1)(2)(2n+1)(2n+3)}{(3)(5)(n)(n+1)} \underset{n\to\infty}{\longrightarrow} \frac{8}{15}.$$

10. What are the last two digits of the number 2019^{2019}?

Answer. 79

Solution. Using the Binomial Theorem, we can write

$$\begin{aligned} 2019^{2019} = -(1-2020)^{2019} &= -(1 - \binom{2019}{1}2020 + \text{terms divisible by } 100) \\ &\equiv (19)(20) - 1 \pmod{100} \equiv 79 \pmod{100}. \end{aligned}$$

11. We pick a random point (x,y,z) with integer coordinates from the cube $S = \{(x,y,z) : |x|,|y|,|z| \le 10\}$. What is the probability of the event that

$$1/x + 1/y + 1/z = 1/(x+y+z)?$$

Answer. 380/3087

Solution. There is a total of $(2\times 10+1)^3 = 21^3$ points with integer coordinates in S. The favorable ones satisfy the equation

$$1/x + 1/y + 1/z - 1/(x+y+z) = (x+y)\left[\frac{1}{xy} + \frac{1}{z(x+y+z)}\right] = 0.$$

Observing that

$$\frac{1}{xy} + \frac{1}{z(x+y+z)} = \frac{xy + z(x+y) + z^2}{xyz(x+y+z)} = \frac{(y+z)(z+x)}{xyz(x+y+z)},$$

the desired equation is equivalent to

$$(x+y)(y+z)(z+x) = 0$$

for nonzero values of x, y, z. Therefore, the subset of favorable points in S is $A \cup B \cup C$, where

$$\begin{aligned} A &= \{(x,y,z) \in S \mid x+y=0,\ xz \neq 0\} \\ B &= \{(x,y,z) \in S \mid y+z=0,\ xy \neq 0\} \\ C &= \{(x,y,z) \in S \mid z+x=0,\ yz \neq 0\} \end{aligned}$$

Using the Exclusion-Inclusion Principle, and applying symmetry we have

$$|A \cup B \cup C| = 3|A| - 3|A \cap B| + |A \cap B \cap C|.$$

It is clear that $|A| = 20^2$, $|A \cap B| = 20$, and $|A \cap B \cap C| = 0$, so $|A \cup B \cup C| = 3(20)^2 - 3(20) = 1140$. The probability of the favorable event, therefore, is equal to $1140/21^3 = 380/3087$.

12. Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) \cdot R(x)$?

Answer. 22

Solution. We can write the problem as

$P(Q(x)) = (Q(x)-1)(Q(x)-2)(Q(x)-3) = P(x) \cdot R(x) = (x-1)(x-2)(x-3) \cdot R(x)$.

Since $\deg P(x) = 3$ and $\deg R(x) = 3$, $\deg P(x) \cdot R(x) = 6$. Thus, $\deg P(Q(x)) = 6$, so $\deg Q(x) = 2$. On the other hand,

$$P(Q(1)) = (Q(1)-1)(Q(1)-2)(Q(1)-3) = P(1) \cdot R(1) = 0,$$
$$P(Q(2)) = (Q(2)-1)(Q(2)-2)(Q(2)-3) = P(2) \cdot R(2) = 0,$$
$$P(Q(3)) = (Q(3)-1)(Q(3)-2)(Q(3)-3) = P(3) \cdot R(3) = 0.$$

Hence, we conclude $Q(1)$, $Q(2)$, and $Q(3)$ must each be 1, 2, or 3. Since a quadratic is uniquely determined by its values at three points, there can be $3 \times 3 \times 3 = 27$ different quadratics $Q(x)$ after each of the values of $Q(1)$, $Q(2)$, and $Q(3)$ are chosen. However, we have included $Q(x)$ which are not quadratic, but linear. Namely,

$$Q(1) = Q(2) = Q(3) = 1 \Rightarrow Q(x) = 1,$$
$$Q(1) = Q(2) = Q(3) = 2 \Rightarrow Q(x) = 2,$$
$$Q(1) = Q(2) = Q(3) = 3 \Rightarrow Q(x) = 3,$$
$$Q(1) = 1, Q(2) = 2, Q(3) = 3 \Rightarrow Q(x) = x,$$
$$Q(1) = 3, Q(2) = 2, Q(3) = 1 \Rightarrow Q(x) = 4 - x.$$

Clearly, we could not have included any other constant functions. For any linear function, we have $2 \cdot Q(2) = Q(1) + Q(3)$ because $Q(2)$ is the y-value of the midpoint of $(1, Q(1))$ and $(3, Q(3))$. So we have not included any other linear functions. Therefore, the desired answer is $27 - 5 = 22$.

13. Suppose that each of 2019 people knows exactly one piece of information, and all 2019 pieces are different. Every time person "A" phones person "B", "A" tells "B" everything he knows, while "B tells "A" nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

Answer. 4036

Solution. Denote the 2019 people by $p_1, p_2, \ldots, p_n$ and a call from p_i, to p_j by $p_i \to p_j$. Let A denote the minimum number of calls which can leave everybody fully informed. The particular sequence

$$p_1 \to p_{2019}, \quad p_2 \to p_{2019}, \quad \ldots, \quad p_{2018} \to p_{2019},$$

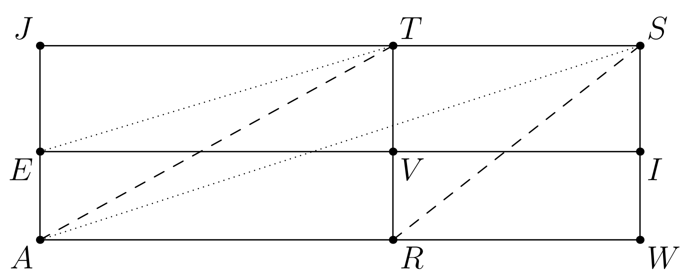
$$p_{2019} \to p_{2018}, \quad p_{2019} \to p_{2017}, \quad \ldots, \quad p_{2019} \to p_1.$$

contains $2(2018) = 4036$ calls and leaves everybody informed, showing that $A \le 4036$. Suppose we have a sequence of calls which leaves everybody fully informed. Consider the "crucial" call at the end of which the receiver (call him p) is the first to be fully informed. Clearly, each of the 2018 people other than p must have placed at least one call no later than the crucial call. Also, each of these 2018 people (being not fully informed) must receive at least one call after the crucial one. Hence the given sequence contains at least 2(2018) calls. Thus $A = 4036$.

14. Point V lies in the interior of rectangle $JAWS$. The feet of the altitudes from V onto sides $\overline{SJ}$, $\overline{WS}$, $\overline{AW}$, and $\overline{JA}$ are T, I, R, E respectively. Given that $[EAST] = 127$, $[TEAR] = 160$, and $[STAR] = 187$, compute $[JARVIS]$. *(The square brackets denote area.)*

Answer. 304

Solution. First, it is incredibly important to draw a picture:



Now, consider the quadrilaterals we are given. First, we have

$$[EAST] = [JAS] - [JET] = \frac{1}{2}[JAWS] - \frac{1}{2}[JEVT] = 127$$

And as for $STAR$, note that it is a trapezoid with its bases summing to the length of JS. Thus, its height is the same height as $JAWS$, and its average base length is half the length of $JAWS$. Thus $[STAR] = \frac{1}{2}[JAWS] = 187$, so $[JAWS] = 374$. Then we deduce that $[JEVT] = [JAWS] - 254 = 120$. Finally,

$$[TEAR] = [RVEA] + [VET] = [RVEA] + 60 = 160$$

and so $[RVEA] = 100$.

Now, we have $[RTJA] = [JEVT] + [RVEA] = 220$, and so $[RTSW] = [JAWS] - [RTJA] = 154$. Using area ratios now, we have

$$\frac{[RVIW]}{[RTSW]} = \frac{[RVEA]}{[RTJA]} = \frac{RV}{RT}$$

and this ratio is $\frac{100}{220} = \frac{5}{11}$. Then $[RVIW] = \frac{5}{11}[RTSW] = 70$. To finish, we have $[JARVIS] = [JAWS] - [RVIW] = 374 - 70 = 304$.

15. Find the sum of the series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right).$$

Answer. $\pi/4$

Solution. Notice that

$$\frac{1}{2n^2} = \frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}.$$

Using the trigonometric identity

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

we have

$$\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1),$$

hence the N-th partial sum of the series after telescoping is given by

$$\sum_{n=1}^{N} (\arctan(2n+1) - \arctan(2n-1)) = \arctan(2N+1) - \arctan(1) \underset{N\to\infty}{\longrightarrow} \frac{\pi}{2} - \frac{\pi}{4}.$$

16. Find the smallest natural number n such that $2^n + 3^n$ is divisible by 125.

Answer. 25

Solution. First, notice that n cannot be even since we would have

$$2^n + 3^n \equiv 2^n + (-2)^n \equiv 2^{n+1} (\bmod 5),$$

which means $2^n + 3^n$ is not even divisible by 5. It is clear that $n = 1$ does not work, so for $n \geq 3$ we have

$$2^n + 3^n = 2^n + (5-2)^n = 2^n + \left[(-2)^n + \binom{n}{1}5(-2)^{n-1} + \binom{n}{2}5^2(-2)^{n-2} + \text{terms divisible by } 5^3\right],$$

so for odd n we have

$$2^n + 3^n \equiv 5n(2^{n-1}) - (25)\frac{n(n-1)}{2}(2^{n-2}) \equiv 2^{n-3}(5n)(9-5n) \pmod{125},$$

which is divisible by 125 if and only if n is divisible by 25 due to the fact that the 2^{n-3} and 9 are not divisible by 5. The smallest such n, therefore, is 25.

17. Compute

$$\int_0^{\sqrt{\pi}} (4x^4+3)\sin(x^2)\, dx.$$

Answer. $2\pi\sqrt{\pi}$

Solution. Consider evaluating the second term by parts, letting $u = \sin(x^2)$, $dv = dx$. Then $v = x$ and $du = 2x\cos(x^2)\,dx$, so

$$\int_0^{\sqrt{\pi}} \sin(x^2)\,dx = x\sin(x^2)\Big|_{x=0}^{\sqrt{\pi}} - 2\int_0^{\sqrt{\pi}} x^2\cos(x^2)\,dx$$

The first term is zero, since $\sin 0 = \sin\pi = 0$. Thus we have

$$\int_0^{\sqrt{\pi}} \sin(x^2)\,dx = -2\int_0^{\sqrt{\pi}} x^2\cos(x^2)\,dx$$

Integrate by parts a second time. We have $u = \cos(x^2)$, $dv = x^2\,dx$. Then $v = \frac{1}{3}x^3$ and $du = -2x\sin(x^2)\,dx$. Thus

$$\int_0^{\sqrt{\pi}} x^2\cos(x^2)\,dx = \frac{1}{3}x^3\cos(x^2)\Big|_{x=0}^{\sqrt{\pi}} + \frac{2}{3}\int_0^{\sqrt{\pi}} x^4\sin(x^2)\,dx = -\frac{\pi\sqrt{\pi}}{3} + \frac{2}{3}\int_0^{\sqrt{\pi}} x^4\sin(x^2)\,dx$$

Thus

$$\int_0^{\sqrt{\pi}} 3\sin(x^2)\,dx = -6\int_0^{\sqrt{\pi}} x^2\cos(x^2)\,dx = 2\pi\sqrt{\pi} - \int_0^{\sqrt{\pi}} 4x^4\sin(x^2)\,dx,$$

which implies

$$\int_0^{\sqrt{\pi}} (4x^4+3)\sin(x^2)\,dx = 2\pi\sqrt{\pi}.$$

The coefficients in this problem have been chosen very specifically so that the terms involving $\int \sin(x^2)\,dx$ cancel out, as this integral has no elementary antiderivative. In particular, many similar integrals such as

$$\int_0^{\sqrt{\pi}} (4x^4-3)\sin(x^2)\,dx$$

have no closed form.

18. Find the multiplicity of 2 in $A = \lfloor (1+\sqrt{3})^{2019} \rfloor$, i.e., find a whole number k such that A is divisible by 2^k, but not by 2^{k+1}. Here, $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.

Answer. 1010

Solution. Using the Binomial Theorem, or by simple induction, one can show that

$$(1+\sqrt{3})^{2019} = m + n\sqrt{3}, \qquad (1-\sqrt{3})^{2019} = m - n\sqrt{3},$$

for some natural numbers m, n. Therefore, the number

$$C := (1+\sqrt{3})^{2019} + (1-\sqrt{3})^{2019} = 2m$$

is an (even) natural number. On the other hand, $-1 < (1-\sqrt{3})^{2019} < 0$, hence $A = C = 2m$. We then make the following observation based on the equality $(1\pm\sqrt{3})^2 = 2(2\pm\sqrt{3})$:

$$\begin{aligned} A &= \frac{1}{1+\sqrt{3}}(1+\sqrt{3})^{2020} + \frac{1}{1-\sqrt{3}}(1-\sqrt{3})^{2020} \\ &= 2^{1009}\left[(\sqrt{3}-1)(2+\sqrt{3})^{1010} - (\sqrt{3}+1)(2-\sqrt{3})^{1010}\right] \\ &= 2^{1009}\left[\sqrt{3}\left((2+\sqrt{3})^{1010} - (2-\sqrt{3})^{1010}\right) - \left((2+\sqrt{3})^{1010} + (2-\sqrt{3})^{1010}\right)\right]. \end{aligned}$$

Simple induction again (or the Binomial Theorem) shows that there are natural numbers u, v such that

$$(2+\sqrt{3})^{1010} = u + v\sqrt{3}, \qquad (2-\sqrt{3})^{1010} = u - v\sqrt{3},$$

which implies

$$A = 2^{1009}\left[\sqrt{3}\left(2v\sqrt{3}\right) - (2u)\right] = 2^{1010}(3v-u).$$

We claim that $3v-u$ is an odd number, and therefore the multiplicity of 2 in A must be 1010. Notice that we have

$$(2+\sqrt{3})^{1010}(2-\sqrt{3})^{1010} = 1 = (u+v\sqrt{3})(u-v\sqrt{3}) = u^2 - 3v^2,$$

which implies that the numbers u and v are of different parity (one is even and one is odd). Therefore, $3v-u$ must be odd.

19. Find $\lim_{n\to\infty} \dfrac{n}{(1\times 2^2 \times 3^3 \times \cdots \times n^n)^{2/n^2}}$.

Answer. $\sqrt{e}$

Solution. Let a_n denote the sequence defined in the problem, and notice that

$$\log a_n = \log n - \frac{2}{n^2}\sum_{k=1}^{n} k \log k.$$

Now, consider the function f defined over $[0,\infty)$ by $f(x) = 2x\log x$ for $x > 0$ and $f(0) = 0$. Notice that f is continuous since $f(0^+) = 0$. In particular, we have

$$\int_0^1 f(x)\,dx = \lim_{t\to 0^+} \int_t^1 2x\log x\,dx$$

Integration by parts with $u = \log x$ and $dv = 2x\,dx$ shows that

$$\int_t^1 x\log x\,dx = \left(x^2\log x\right)\big|_t^1 - \int_t^1 x\log x\,dx = -t^2\log t - \frac{1-t^2}{2}.$$

Therefore, by taking the limit as $t \to 0^+$ we get

$$\int_0^1 f(x)\,dx = -\frac{1}{2}.$$

Since f is Riemann integrable on $[0,1]$, we can take a uniform partition with n subintervals and choose their right endpoints k/n for $k = 1, 2, \ldots, n$ and form the Riemann sum, which tends to the integral of f over the interval, in other words:

$$\int_0^1 f(x)\,dx = -\frac{1}{2} = \lim_{n\to\infty} \frac{1-0}{n}\cdot\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\frac{2k}{n}\log\left(\frac{k}{n}\right)$$

This implies

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}(2k\log k - 2k\log n) = -\frac{1}{2},$$

or, in other words, after multiplying by -1 and summing up the $2k$'s we have

$$\lim_{n\to\infty}\left(\frac{n(n+1)}{n^2}\,\log n - \frac{2}{n^2}\sum_{k=1}^{n} k\log k\right) = \frac{1}{2}.$$

This means

$$\lim_{n\to\infty}\left(\log a_n - \log n + \frac{n(n+1)}{n^2}\,\log n\right) = \frac{1}{2}.$$

But we have

$$\lim_{n\to\infty}\left(-\log n + \frac{n(n+1)}{n^2}\,\log n\right) = \lim_{n\to\infty}\frac{\log n}{n} = 0,$$

hence $\lim_{n\to\infty}\log a_n = 1/2$ and $\lim_{n\to\infty} a_n = e^{1/2}$.