EF Exam
Texas A&M High School Math Contest
November 9, 2019

All answers must be simplified, and if units are involved, be sure to include them.

1. Find the minimum value of m for which the equation $x^2 + 2mx + 3m^2 + m - 21 = 0$ has real roots.

2. Let (x, y) be a solution of the system

$$\begin{cases} 25^x \cdot 125^y &= 1 \\ 1 \div 9^{3y} &= 81\sqrt{3}(\sqrt{3})^x. \end{cases}$$

   Find $x + y$.

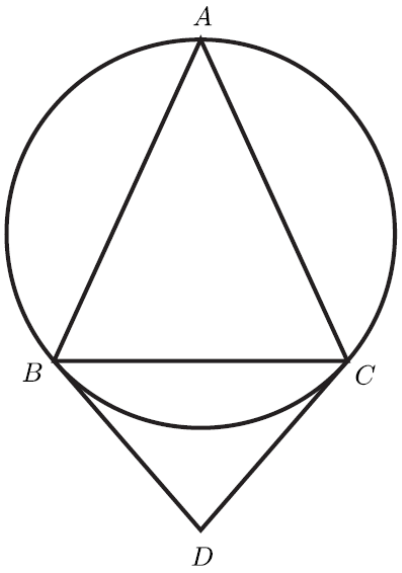
3. Let a and b be the solutions of the equation $x^2 - 6x + 4 = 0$. Find the value of

$$(a^{2019} + b^{2019}) - 6(a^{2018} + b^{2018}) + 4(a^{2017} + b^{2017}) + a^2 + b^2.$$

4. Let $f(x) = 4x^3 - 5x^2 + px + q$, where p and q are integers and suppose that $x^2 + 3x - 4$ is a factor of $f(x)$. Find pq.

5. In the expansion of $(1 + ax - x^2)^8$ where a is a positive constant, the coefficient of x^2 is 244. Find the value of a.

6. An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.



7. Let $P(x) = (5x^3 + 2x^2 - 4x + 6)^4$. If $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, find $a_1 + a_3 + a_5 + \cdots$.

8. Find $\lim\limits_{n\to\infty} x_n$ where

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right), \quad n \geq 2.$$

9. Let $f(x) = 4x^m + 5x^{-n}$, where m and n are positive integers. If $x^2 f''(x) + 2xf'(x) = 6f(x)$ find $m + n$.

10. Solve the equation

$$(x + 1)^{\log_3(x-2)} + 2(x - 2)^{\log_3(x+1)} = 3x^2 + 6x + 3.$$

11. Any five points are taken inside or on a square of side 1. Find the smallest possible number a such that it is always possible to select one pair of points from these five such that the distance between them is equal to or less than a.

12. Find the product of all the solutions in $[0, 2\pi)$ of the inequality

$$\sin 4x - \sqrt{2}\cos\left(4x - \frac{\pi}{4}\right) \geq 1.$$

13. Find the exact value of the integral $\displaystyle\int_0^1 x\ln(x+1)dx$.

14. It is given that $\sin\theta + \cos\theta = \sqrt[4]{3}$. Find the exact value of $\sin^5\theta + \cos^5\theta$.

15. Find the exact value of the expression

$$\sin 1^\circ\left(\frac{1}{\cos 0^\circ\cos 1^\circ} + \frac{1}{\cos 1^\circ\cos 2^\circ} + \cdots + \frac{1}{\cos 59^\circ\cos 60^\circ}\right).$$

16. Find the largest real solution of the equation

$$(x-1)(x-3)(x-5)(x-7)(x-9)(x-11) = -225.$$

17. Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 x dx$. (Hint: Differentiate the function $\sin^{n-1} x\cos x$.)

18. Consider the sequence $(a_n)_{n\geq 1}$, with

$$a_n = \lim_{x\to 0}(1 - x\sin nx)^{1/x^2}.$$

Find $\displaystyle\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n)$.

19. Simplify $\arctan\frac{1}{1+1+1^2} + \arctan\frac{1}{1+2+2^2} + \arctan\frac{1}{1+3+3^2} + \cdots + \arctan\frac{1}{1+n+n^2}$.

20. Find the value of the limit

$$L = \lim_{x\to\infty}\int_0^x \frac{1}{(1+t^2)(1+t^4)}dt.$$