2021 Power Team
Texas A&M High School Mathematics Contest
Solutions

Problem 1. Alice chose an integer from 1 to 100. Bob wants to find this number by asking Alice "yes or no" questions. Find the smallest number of questions Bob can ask Alice so that he will for sure guess the number chosen by Alice.

Each answer has 2 outcomes (possible answers), so n question will have 2^n possible outcomes, so they can distinguish at most 2^n cases. It follows that 6 questions will be not enough, as they will distinguish $2^6 = 64$ cases (i.e., there are 64 possible answers to 6 "yes or no" questions). On the other hand, 7 questions should be enough. Namely, the first question can be "Is the number greater than 50?" The answer will tell us in which of the two intervals $[1, 50]$ or $[51, 100]$ is Alice's number. Let $[a, b]$ be the corresponding interval. The second question "Is the number greater than $(a+b)/2$?" The answer to the second question will tell us in which of the four intervals $[1, 25], [26, 50], [51, 75], [76, 100]$ is Alice's number. If $[a, b]$ is the corresponding interval, then Bob asks "Is the number greater than $(a+b)/2$?", and so on. If the interval $[a, b]$ contains k integers, then after the answer to this question, Bob will know that the number belongs to an interval of length $k/2$, $(k-1)/2$ or $(k+1)/2$. It follows that after 7 answers the interal will contain at most $(((((25+1)/2+1)/2+1)/2+1)/2+1)/2 = 25/32+1/32+1/16+1/8+1/4+1/2 = (25+1+2+4+8+16)/32 < 2$ numbers. Consequently, Bob will know the answer not later than after the seventh question.

Problem 2. Suppose that we have 9 coins: 8 of the same weight, and one fake coin visually indistinguishable for the other 8 but lighter. What is the smallest number of weighings on a balance scale needed to find the fake coin?

Let us split the coins into three piles of 3 coins each, and compare two piles. If one of them is lighter than the other, then we know that the fake coin is the lighter pile. If they have the same weight, then the fake coin is in the third pile. In any case, we will find three coins one of which is fake. Then we compare two coins out of these three. If one coin is lighter, then we know that it is the fake coin. If they have equal weight, then the thrid coin is fake. Consequently, two weighings is enough.

One weighing gives one of three results: the pile on the left pan is lighter, the pile on the right pan is lighter, or the piles on both pans have equal weight. Consequently, one weighing can distinguish at most three cases. We are asked to distinguish 9 cases (which one of the nine coins is fake). It follows that we can not do it in one weighing. So, two weighings is the minimal number.

Problem 3. Find a formula for the number of weighings needed to find the fake coin if there are $N - 1$ coins of one weight and one lighter fake coin.

If we do k weighings, then we can distinguish 3^k cases. Consequently, if $3^k < N$, we will not be able to find the fake coin in k weighings in all cases. Let k be the smallest number such that $3^k \geq N$. In other words, $k = \lceil \log_3 N \rceil$.

Let us split the coins into three almost equal piles. If N is divisible by 3, then we split into three equal piles. If $N = 3m + 1$, then we split them into two piles of m coins in each and one pile of $m + 1 = (N + 2)/3$ coins. If $N = 3m + 2$, then split into two piles of $m + 1 = (N + 1)/3$ coins in each and one pile of m coins. Put two piles with equal number of coins on the pans of the scale. If one of the piles is lighter, then we know that the fake coin is in the lighter pile. Otherwise, it is in the third pile. We will reduce our pile to a pile of at most $(N + 2)/3$ coins containing the fake coin.

If $N < 3^k$, then after k weighings our pile containing the fake coin will have at most

$$\frac{N}{3^k} + \frac{2}{3^k} + \frac{2}{3^{k-1}} + \cdots + \frac{2}{3} = \frac{N}{3^k} + 1 - \frac{1}{3^k} \leq 2 - \frac{1}{3^k} < 2$$

coins, so we will also find the fake coin.

Therefore, the answer is: the smallest number of weighings we need to find the coin is $\lceil \log_3 N \rceil$.

Problem 4. There are 8 coins. We know that either all of them have the same weight, or one is fake (lighter). What is the smallest number of weighings you need to find the fake coin or to prove that it does not exist?

There are 9 cases: one case when all coins are genuine, and 8 cases depending on which of the 8 coins is fake. We can reduce therefore the problem to Problem 2 by adding one "imaginary coin."

More explicitly, we split 8 coins into two piles of 3 coins each and one pile of 2 coins (3 together with the "imaginary coin"). We compare the weights of the two piles of 3 coins. If one of them is lighter, then we know that the lighter pile contains the fake coin, so weighing two of them we will find it, as in Problem 2.

If the two piles of 3 coins have equal weight, then we know that either there is no fake coin, or one of the two remaining coins is fake. Comparing these two coins, we will either find a lighter one, i.e., the fake coin, or we will see that they are of equal weight, so there was no fake coin (the "fake coin" was imaginary).

One weighing is not enough, since it will distinguish only 3 cases instead of 9.

Problem 5. We have one golden coin, 3 silver coins, and 5 brass coins. Coins made of different metals are visually different and have different weight. One of the coins is fake (lighter than its legal weight). What is the smallest number of weighings needed to find the fake coin?

Again, as in the previous problem, there are 9 cases, so the minimal number of weighings is at least 2.

Let us take for the first weighing 1 silver coin and 2 brass coins on both pans. If the scale is not in equilibrium, we will know that the fake coin is in the lighter pan, so it is either a silver coin or one of the two brass coins. In this case, we compare the two suspicious brass coins. If one of them is lighter, it is fake. If they have equal weight, then the silver coin (which was together with them during the first weighing) is fake.

If the scale is in equilibrium, then we know that the fake coin is among the remaining 1 silver, 1 brass, and 1 golden coin. Let us put the silver and the brass coin on different pans of the scale, and add 1 brass coin to the silver coin, and one silver coin to the brass coin from the coins that were weighed the first time (we know that all of them are genuine). If the scale is in equilibrium, then the golden coin is fake. Otherwise, we will see which of the pairs of 1 silver and 1 brass coins contains the fake one. Since we know which one of them is for sure genuine, we will know which one is fake.

Problem 6. We have 12 golden coins and 12 silver coins. One of the coins is fake. A fake silver coin is lighter than a genuine silver coin. A fake golden coin is heavier than a genuine golden coin. You are not allowed to put more than 4 coins of the same metal on one pan of the scale. What is the smallest number of weighings needed to find the fake coin?

We have 24 cases. Consequently, we need at least 3 weighings (since 2 weighings will distinguish at most 9 cases).

For the first weighing, we put 4 golden and 4 silver coins on both pans. If the scale is in equilibrium, we know that the fake coin is among the remaining 4 golden and 4 silver coins (the ones that were not on the scale). If they are not in equilibrium, the fake coin is either one of the 4 golden coins in the heavier pile or one of the 4 silver coins in the lighter pile.

Consequently, after the first weighing we will find 4 golden and 4 silver coins such that the fake coin is one of them. The remaining coins are genuine.

Let us put then on the scale for the second weighing the 3 suspicious golden and the 3 suspicious silver coins on one pan, and 3 genuine golden and 3 genuine silver coins on the second pan. If the scale is in equilibrium, then we will know that the fake coin is one of the remaining 1 golden and 1 silver suspicious coins. Then we can compare one of them (e.g., the golden coin) with a genuine coin, and will know if it is fake, or if the other coin is fake.

If the scale is not in equilibrium, then we will know if the fake coin is lighter (i.e., if it is silver) or heavier (i.e., golden). Therefore, we will have only three suspicious coins of the same metal. Weighing two of them will determine the fake coin.

We see that following the above algorithm we will find the fake coin in three weighings.

Problem 7. We have 9 indistinguishable coins, one of which is fake (lighter than the other coins). We have three balance scales, one of which is broken (it produces random unpredictable results not related to the actual weights on the pans). We don't know which coin is fake and which scale is broken. Find the fake coin by 4 weighings.

Let us arrange the coins in a square

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

Let us compare two columns on the scale number 1, and two rows on the scale number 2. According to the solution of Problem 2, this will give us one column and one row containing a fake coin. Since we don't know if and which of the two scales is broken, we get five suspicious coins: the union of one row and one column. Let us assume that the suspicious coins are in the first row and the first column, that is, $\{1, 2, 3, 4, 7\}$ (if not, then just rename them).

Let us compare the coins $\{2, 3\}$ and the coins $\{4, 7\}$ on the third scale. If the third scale is not broken, and it is in equilibrium, coin number 1 is fake. If the third scale is not broken, and is not in equilibrium, then we know two suspicious coins and we know which of the first two scales is broken (the one which made a wrong prediction).

If the third scale is broken, then the first two are not, so the fake coin is number 1.

Consequently, if the result of the third weighing is equality, then either the scale number 3 is broken and the fake coin is number 1, or the third scale is not broken and the fake coin is number 1. So, in any case, the fake coin is number 1.

If the result of the third weighing is inequality, then either the third scale is broken, the first two scales are not broken, and the fake coin is 1; or the third scale is not broken, we have two suspicious coins, and we know which of the two first scales is not broken. So, in any case, we have three suspicious coins, and we know one scale which is for sure not broken. So, we can use this working scale to select which of the three coins is fake, as in Problem 2.

Problem 8. We have 20 visually indistinguishable coins of two kinds: x coins of weight a each and $20 - x$ coins of weight b each, where $b > a$. Find x using 11 weighings.

Take two coins and compare them. If they have different weight, then they are of two kinds. Take them, split the remaining 18 coins into groups of 2, and compare each pair with the two selected coins. If a pair is heavier, then you know that they are both of weight b. If it is lighter, then it consists of two coins of weight a. If it has the same weight as the selected pair, then one of them is of weight a, and the other of weight b. So, in any case you will know how many coins of each type is there in each pair, and after 10 weighings you will know x.

Suppose now that the selected coins have equal weight. Then they are both of the same kind, but we do not know which. Let us still compare the selected pair with the 9 remaining pairs. As soon as we get an inequality of the weight of the selected pair and a tested pair, we will know the type of the selected pair, and hence the type of all previous pairs which had equal weight with the selected pair. Then we can compare the two coins of the tested pair, we will know their type and the type of all previously weighed coins. After that we can proceed as in the previous case. We will get 11 weighings in total in this case.

Problem 9. We have 13 masses of 1, 2, 3, …, 13 oz (labeled by their weights). But exactly one of the masses is defective (its weight is different from the label, but it is not known if it is heavier or lighter). What is the smallest number of weighings needed to find the wrong mass?

Let us do the following weighings:

$1+2+3+4+5+8$ vs $6+7+10$

$1+2+3+6+9+10$ vs $8+11+12$

$1+5+7+9+12$ vs $3+8+10+13$

If 1 is wrong, then we get $>, >, >$ or $<, <, <$. If 2 is wrong, then we get $>, >, =$ or $<, <, =$.
If 3 is wrong, then we get $<, <, >$ or $>, >, <$.
If 4 is wrong, then we get $>, =, =$ or $<, =, =$.
If 5 is wrong, then we get $<, =, <$ or $>, =, >$.
If 6 is wrong, then we get $<, >, =$ or $>, <, =$.
If 7 is wrong, then we get $<, =, >$ or $>, =, <$.
If 8 is wrong, then we get $<, >, >$ or $>, <, <$.
If 9 is wrong, then we get $=, <, <$ or $=, >, >$.
If 10 is wrong, then we get $<, >, <$ or $>, <, >$.
If 11 is wrong, then we get $=, <, =$ or $=, >, =$.
If 12 is wrong, then we get $=, <, >$ or $=, >, <$.
If 13 is wrong, then we get $=, =, <$ or $=, =, >$.

We see that the results of the weighings uniquely determine the wrong mass. Two weighings will distinguish at most 9 cases, which is not enough.

Problem 10. We have 10 masses of 1, 2, 3, …, 10 oz, labeled by stickers with their weights written on them. However, two stickers for masses different by 1 oz (i.e., n and $n+1$ for some n) were switched. What is the smallest number of weighings needed to find switched stickers?

We need to find one case out of 9, so we need at least two weighings.

At first, let us compare the masses $3+5$ vs 8. If we get equality, then the pairs of switched labels can be $(1,2), (6,7)$ or $(9,10)$. If $3+5>8$, then the pairs of switched labels are $(3,4)$, $(5,6)$ or $(7,8)$. If $3+5<8$, then it is $(2,3), (4,5)$ or $(8,9)$.

Second time we can, for example compare $2 + 5$ and 7, and distinguish all cases.

Namely, in the first case, $(1, 2)$ will produce $2 + 5 < 7$, $(6, 7)$ will produce $2 + 5 > 7$, and $(9, 10)$ will give $2 + 5 = 7$.

In the second case, $(3, 4)$ will give $2 + 5 = 7$, $(5, 6)$ will give $2 + 5 > 7$, and $(7, 8)$ will give $2 + 5 < 7$.

In the third case $(2, 3)$ will give $2 + 5 > 7$, $(4, 5)$ will give $2 + 5 < 7$, and $(8, 9)$ will give $2 + 5 = 7$.