DE Exam
Texas A&M High School Math Contest
November 12, 2022

Directions: *All answers should be simplified.*

1. It is known that $(\sqrt{2}-1)^4 = \sqrt{N} - \sqrt{N-1}$, where N is an integer. Find N.

2. A parallelogram has sides of length 2 and 3. One of its diagonals has length 4. Find the length of the other diagonal.

3. Find the probability that an integer number chosen randomly in the range from 1 to 10^5 (inclusive) has exactly three odd digits (not necessarily distinct).

4. Evaluate the following expression: $\cos\left(\arcsin\frac{4}{5}\right) + \arccos\left(\sin\frac{4}{5}\right)$.

5. Find the largest integer x satisfying the inequality $x < \sqrt{|x-2022|}$.

6. Suppose T is a triangle in the plane with sides of length 2, 3 and 4. Let F be the figure that consists of all points of T as well as all points at distance at most 1 from the triangle. Find the perimeter of the figure F.

7. A function $f : (0,\infty) \to (0,\infty)$ satisfies a functional equation $x + f(x) = 2f(1/x)$ for all $x > 0$. Find $f(3)$.

8. Find the shortest distance between two circles in the coordinate plane given by equations $x^2 + y^2 = 81$ and $x^2 + y^2 + 6x - 8y + 21 = 0$.

9. How many real solutions does the equation $(x^2 - x - 1)^{5x^2-19x+16} = 1$ have?

10. Let P be a pentagon in the coordinate plane with vertices at points $(0,0)$, $(4,0)$, $(5,2)$, $(3,4)$ and $(-1,2)$. Find the area of P.

11. Let r be a real root of the equation $x^3 - x + 1 = 0$. Evaluate the expression $r^5 + r^4 + r^2 + \frac{1}{r}$.

12. Suppose S_1 and S_2 are two circles of radius 1 that touch each other at the point O. Let S be a circle centered at O and tangent to both S_1 and S_2. Let S_0 be a circle that touches S internally and touches S_1 and S_2 externally. Find the radius of S_0.

13. Let f be the function of a real variable given by the formula $f(x) = \dfrac{cx + c^2}{3x - 6}$, where c is a real number. Determine all values of the parameter c for which the function f is invertible and, moreover, coincides with its inverse function on the intersection of their domains.

14. Let P be a regular triangular pyramid (that is, the base of P is an equilateral triangle and the apex is projected onto the center of the base). It is given that three edges of the pyramid have length 4 and the other three edges have length 7. Find the volume of P.

15. Find a triple of integers (a, b, c) such that $90 < a < b < c < 180$ and the sum of any two of the numbers a, b, c is a perfect square.

16. A regular dodecagon (12-gon) is inscribed into a circle of radius 1. How many diagonals of the dodecagon intersect the concentric circle of radius $1/3$?

17. Find the sum of all even integers n in the range from 1 to 400 such that the sum $1 + 2 + 3 + \cdots + n$ is a perfect square.

18. In a triangle ABC with $|AB| > |AC|$, the median AM, the angle bisector AD and the altitude AH divide the angle BAC into 4 equal parts. Find $\angle ABC$.

19. The number $5 \cdot 3^2 \cdot 2^{336}$ is the smallest positive integer that has exactly 2022 different divisors. How many digits does this number have when written out (in decimal notation)?

20. Find the least positive integer n such that the sum

$$\sin 14^\circ + \sin 28^\circ + \sin 42^\circ + \cdots + \sin(14n)^\circ$$

has negative value.