# Solutions to AB Exam

Texas A&M High School Math Contest
4 November, 2023

1. Find the exact value of $\dfrac{77.85 \times \dfrac{3217}{100} - \dfrac{7785}{100} \times 11.94}{20.23 \times \dfrac{832}{1000} + \dfrac{668}{1000} \times 20.23}$.

   The fraction is equivalent to $\dfrac{(77.85)(32.17) - (77.85)(11.94)}{(20.23)(0.832) + (0.668)(20.23)} = \dfrac{(77.85)(20.23)}{(20.23)(1.5)} = \dfrac{77.85}{1.5} = \mathbf{51.9}$

2. There are 120 different 5-digit numbers that use each of the digits 2, 4, 5, 7, and 8 exactly once. If listed in numerical order from smallest to largest, what number is in the 97th position of the list?

   There are 24 numbers that start with each digit, so the first 96 numbers consist of all the numbers which start with 2, 4, 5, and 7. Therefore, the 97th number in the list is the smallest number which starts with 8, or **82,457.**

3. A resale store sells certain items at a price 50% below the original price. On Saturdays, these items are discounted an additional 20% off the sale price. If you purchase one of these items on a Saturday, what percentage of the original price do you pay?

   Let $P$ be the original price of the item. After the first discount, the price is $\frac{1}{2}P$. On Saturdays, you pay 80%, or $\frac{4}{5}$ of the sale price $= \frac{4}{5}\left(\frac{1}{2}P\right) = \frac{2}{5}P$. So you pay **40%** of the original price.

4. Your friend has a list of seven numbers. They tell you (truthfully) that the average of the first four numbers is 5, the average of the last four numbers is 8, and the average of all seven numbers is $6\frac{4}{7}$. What is the middle number?

   The sum of the first four numbers is $(5)(4) = 20$. The sum of the last four numbers is $(8)(4) = 32$, so the total of the eight numbers (counting the middle number twice) is 52. The sum of all seven numbers is $\left(\frac{46}{7}\right)(7) = 46$, so the middle number is **6.**

5. Write $\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right)\cdots\left(1 - \frac{2}{2022}\right)\left(1 - \frac{2}{2023}\right)$ as a single fraction.

   The expression is equivalent to

   $$\left(\frac{1}{3}\right)\left(\frac{2}{4}\right)\left(\frac{3}{5}\right)\left(\frac{4}{6}\right)\left(\frac{5}{7}\right)\cdots\left(\frac{2020}{2022}\right)\left(\frac{2021}{2023}\right)$$

   Every numerator cancels except the first two, and every denominator cancels except the last two, so our product is

   $$\frac{(1)(2)}{(2022)(2023)} = \frac{\mathbf{1}}{\mathbf{(1011)(2023)}} = \frac{1}{2023000 + 20230 + 2023} = \frac{\mathbf{1}}{\mathbf{2,045,253}}$$

6. BT Parnum has a traveling Brass Band & Rhino Circus. The group involved had a total of 150 legs and 150 horns. If each band member had two legs and packed three horns, and each rhinocerous had four legs and the usual one horn, what is $\frac{R}{B}$, the ratio of the number of rhinocerouses to the number of band members.

   We have the following system of equations: $\begin{array}{c} 2B + 4R = 150 \\ 3B + R = 150 \end{array}$. We can solve for $R$ and $B$, or note that

$$2 + 4\frac{R}{B} = \frac{150}{B}$$
$$3 + \frac{R}{B} = \frac{150}{B}$$

   Subtracting these equations yields $3\frac{R}{B} = 1$, or $\frac{R}{B} = \mathbf{\frac{1}{3}}$.

7. What is the remainder when $(2023)^1 + (2023)^2 + (2023)^3 + \cdots + (2023)^{2022} + (2023)^{2023}$ is divided by 10?

   The remainder divided by 10 is just the last digit, and in general the last digit of $(2023)^n$ =the last digit of $(3)^n$ (mathematically, $2023^n \text{mod } 10 = 3^n \text{mod } 10$). Therefore, our expression mod 10 is equivalent to

$$3 + 9 + 7 + 1 + 3 + 9 + 7 + \cdots + 1 + 3 + 9 + 7$$
$$= (3 + 7) + (9 + 1) + (3 + 7) + (9 + 1) \cdots + (3 + 7) + 9$$

   Since $3 + 7 = 0$ mod10 and $9 + 1 = 0$ mod 10,

$$= 0 + 0 + 0 + 0 + \cdots + 0 + 9$$

   so our remainder is **9.**

8. If $p$ and $q$ are the roots of $x^2 + 2023x + 114$, what is $\frac{1}{p} + \frac{1}{q}$?

   $\frac{1}{p} + \frac{1}{q} = \frac{q+p}{pq}$. Further, we know $(x - p)(x - q) = x^2 - (p + q)x + pq = x^2 + 2023x + 114$, so $pq = 114$ and $p + q = -2023$. Therefore, $\frac{1}{p} + \frac{1}{q} = -\mathbf{\frac{2023}{114}}$.

9. A (really big) gumball machine contains 2023 blue gumballs, 2023 red gumballs, 2023 green gumballs, and 2023 yellow gumballs. If each gumball costs 1 cent, what is the least amount of money you have to spend in order to guarantee you'll have at least 4 gumballs of the same color (any color)?

   The largest amount of money you can spend without having at least 4 gumballs of one color is 12 cents (3 blue, 3 red, 3 green, and 3 yellow). The next gumball guarantees you four of one color, so you need **13 cents**.

10. The sum of two natural numbers $a$ and $b$ is 2023. What is the largest possible value of their greatest common divisor, $gcd(a, b)$?

    If $k$ is a divisor of both $a$ and $b$, then there are natural numbers $c$ and $d$ such that $a = kc$ and $b = kd$, so $2023 = a + b = k(c + d)$. On the other hand, $2023 = 7 \cdot 17^2$, so $k$ must be 1, 7, 17, $(7)(17) = 119$, or $17^2 = 289$. Thus, the largest possible value of the GCD is **289** (possible pairs of numbers $a$ and $b$, in either order, are 289, (289)(6) or (289)(2), (289)(5) or (289)(3), (289)(4) ).

11. Let $x$ be a solution to the equation $\sqrt{3-x} = 1-x$. Find the sum of all possible values of $x$.

    Square both sides and solve the resulting quadratic equation: $3-x = x^2-2x+1 \rightarrow x^2-x-2 = 0$, so $(x-2)(x+1) = 0$ and $x = 2$ or $x = -1$. However, $x = 2$ is not a valid solution since $\sqrt{3-2} \neq 1-2$, so the only solution (hence the sum of all solutions) is $\mathbf{-1}$.

12. Find the smallest real value of $m$ for which the equation $x^2 + 2mx + 3m^2 + m - 21 = 0$ has real solutions for the variable $x$.

    To have real roots, we need the discriminant to be nonnegative: $(2m)^2 - 4(1)(3m^2+m-21) \geq 0$, or $-8m^2 - 4m + 84 \geq 0$, or $2m^2 + m - 21 \leq 0$ . This is true when $m \in \left[-\frac{7}{2}, 3\right]$, so the smallest possible value of $m$ is $-\boldsymbol{\frac{7}{2}}$.

13. The product of three consecutive integers is divided by the first integer. Then the product of the three is divided by the second integer. Finally, the product of the three is divided by the third integer. If the sum of the three quotients is 254 more than three times the square of the smallest integer, what is the smallest integer?

    Let the integers by $n$, $n+1$, and $n+2$. Then $(n+1)(n+2) + (n)(n+2) + n(n+1) = 3n^2 + 254$, so $6n + 2 = 254$, or $n = \mathbf{42}$.

14. You live in a state whose income tax is $p\%$ of the first \$28,000, then $(p+2)\%$ of any amount above \$28,000. Last year, your overall income tax amounted to $(p+0.25)\%$ of your income. What was your income last year?

    Let $I$ be your income. Then $28000\left(\frac{p}{100}\right) + (I - 28000)\left(\frac{p+2}{100}\right) = I\left(\frac{p+\frac{1}{4}}{100}\right)$. Multiplying out and canceling like terms gives us $\frac{2I}{100} - 560 = \frac{I}{400}$, or $I = \mathbf{\$32,000}$.

15. How many positive integers from 1 to 2023 are multiples of 3 or 4, but not multiples of 5?

    **Solution 1:** We start with the numbers from 1 to 2000, the largest common multiple of 3, 4, and 5 less than 2023. Eighty percent of these, or $\frac{4}{5} \cdot 2000 = 1600$, are not multiples of 5. We remove from this group the number which are not multiples of 3 and not multiples of 4, which is $\frac{2}{3} \cdot \frac{3}{4} \cdot 1600 = 800$, leaving 800 numbers which are multiples of 3 or multples of 4. For the remaining numbers, we have 2001, 2004, 2007, 2008, 2012, 2013, 2016, 2019, and 2022 for a total of **809 numbers**.
    **Solution 2:** The general formula for the required number of integers from 1 to n is $(\lfloor n/3 \rfloor + \lfloor n/4 \rfloor - \lfloor n/12 \rfloor) - (\lfloor n/15 \rfloor + \lfloor n/20 \rfloor - \lfloor n/60 \rfloor)$, where $\lfloor x \rfloor$ refers to the largest integer less than or equal to $x$. The first 3 terms give the number of integers from 1 to $n$ which are divisible by 3 or 4 (subtracting the number of integers divisible by 12 as they are counted twice in the first two terms). Similarly, the second three terms give the number of integers divisible by 5 which are also divisible by 3 or 4. For $n = 2023$, we get

    $$(\lfloor 2023/3 \rfloor + \lfloor 2023/4 \rfloor - \lfloor 2023/12 \rfloor) - (\lfloor 2023/15 \rfloor + \lfloor 2023/20 \rfloor - \lfloor 2023/60 \rfloor)$$

    $$= (674 + 505 - 168) - (134 + 101 - 33) = 1011 - 202 = 809$$

16. The sum of the reciprocals of two numbers $x$ and $y$ is 4, and the sum of the squares of the reciprocals is 11. What is the product of the two numbers?

We have $\dfrac{1}{x}+\dfrac{1}{y}=4$ and $\dfrac{1}{x^2}+\dfrac{1}{y^2}=11$. Square the first equation and subtract the second to yield $\dfrac{2}{xy}=5$, so $xy=\mathbf{\dfrac{2}{5}}$.

17. A curve consists of all the points $(x, y)$ which are vertices of a parabola of the form $y = x^2+2ax+a$ for some real number $a$. Find the equation of the curve.

    Completing the square yields $x^2 + 2ax + a = (x + a)^2 + (a - a^2)$, so the vertex is $(-a, a - a^2)$. Therefore, the curve has equation $\boldsymbol{y = -x^2 - x}$.

18. How many pairs of integers $(x, y)$ are solutions to the equation $\sqrt{x}+\sqrt{y}=\sqrt{5290}$?

    We have $\sqrt{x} = \sqrt{5290} - \sqrt{y}$, so squaring both sides of the equation and using the fact that $\sqrt{529} = 23$, we get $x = 5290 + y - 46\sqrt{10y}$. For $x$ to be an integer, we need $y = 10a^2$ for some nonnegative integer $a$. A similar argument shows that $x = 10b^2$ for some nonnegative integer $b$. Therefore, $\sqrt{10b^2}+\sqrt{10a^2}=\sqrt{5290} \to a\sqrt{10}+b\sqrt{10}=23\sqrt{10}$. So we need $a+b = 23$, for which there are **24 solutions** ($a = 0, 1, 2, \cdots 23$ and $b = 23, 22, 21, \cdots 0$).

19. You start with a positive integer $N$ and create a sequence of numbers, where the next number is obtained by subtracting the largest possible perfect square from the current number until 0 is obtained. For example, if you start with 23, your sequence is 23, 7 (23-16), 3 (7-4), 2 (3-1), 1 (2-1), and 0 (1-1), which contains 6 numbers. You manage to find a sequence which contains 2023 numbers. What is the ones digit of the smallest possible starting number $N$?

    We start by building the sequence in reverse: we begin with 0 and $0 + 1^2 = 1$ for the last two numbers. We continue by adding the smallest possible perfect square such that our sum is less than the next perfect square. For instance, our next number is $1 + 1^2 = 2$, then $2 + 1^2 = 3$. However, we cannot add $1^2$ again since that gives us $2^2$, so our next number is $3 + 2^2 = 7$. $7 + 2^2 > 3^2$, and $7 + 3^2 = 4^2$, so our next number is $7 + 4^2 = 23$. More generally, let $a_n$ be the smallest number such that the sequence described in the problem contains $n$ numbers. Then we need $a_{n+1} = a_n + x^2$, where $x$ is the smallest positive integer such that $a_n + x^2 < (x+1)^2$, which means $x > \dfrac{a_n - 1}{2}$. If $a_n$ is odd, then $x = \dfrac{a_n - 1}{2} + 1 = \dfrac{a_n + 1}{2}$, so

    $$a_{n+1} = a_n + \left(\frac{a_n + 1}{2}\right)^2 \quad (1)$$

    In our current sequence, $a_6 = 23$ is odd, so from (1), it follows that $a_7 = 23 + \left(\dfrac{23+1}{2}\right)^2 = 23 + 12^2 = 167$.
    At this point, our "reverse sequence" is $\{0, 1, 2, 3, 7, 23, 167\}$ and we may notice that the ones digit is now alternating between 3 and 7. To prove this observation in general, one can use formula (1) with the following two cases:

    (a) If the ones digit of $a_n$ is 3, then $a_n$ is odd, so from (1) the ones digit of $a_n + 1$ is the same as the ones digit of $3 + \left(\dfrac{3+1}{2}\right)^2 = 7$.

    (b) If the ones digit of $a_n$ is 7, then $a_n$ is odd, so from (1) the ones digit of $a_n + 1$ is the same as the ones digit of $7 + \left(\dfrac{7+1}{2}\right)^2 = 23$, which is 3.
    Using this alternating pattern for $n \geq 6$, the ones digit of $a_n$ is 3 if $n$ is even, and 7 if $n$ is odd. Hence, the ones digit of $a_{2023}$ is **7**.

20. Four sparrows found a dish of seeds
Fine birdie food, was free of weeds
Said Pip: "In turn each take two grains,
Plus a third of what remains.
I'll go first, then Pep, then Pop,
With Pap the last. And then we stop."
But Pap cried out, "It isn't fair.
Mine's two seeds less than half Pep's share."
But Pip was boss; his word was law
So little Pap got nothing more.
Poor Pap, his share was rather small!
How many seeds were there in all?

Let $\boldsymbol{N}$ be the number of seeds in the dish. The chart below shows how many seeds each bird gets and how many remain after each turn:

| Pip | remaining | Pep | remaining |
|---|---|---|---|
| $\frac{1}{3}(N-2)+2$ | $N-\frac{N+4}{3}$ | $\frac{1}{3}\left(\frac{2N-4}{3}-2\right)+2$ | $\frac{2N-4}{3}-\frac{2N+8}{9}$ |
| $=\frac{N+4}{3}$ | $\frac{2N-4}{3}$ | $=\frac{2N+8}{9}$ | $=\frac{4N-20}{9}$ |

| Pop | remaining | Pap |
|---|---|---|
| $\frac{1}{3}\left(\frac{4N-20}{9}-2\right)+2$ | $\frac{4N-20}{9}-\frac{4N+16}{27}$ | $\frac{1}{3}\left(\frac{8N-76}{27}-2\right)+2$ |
| $\frac{4N+16}{27}$ | $\frac{8N-76}{27}$ | $\frac{8N+32}{81}$ |

Therefore, $\frac{8N+32}{81}=\frac{1}{2}\cdot\frac{2N+8}{9}-2$. Clearing the fractions yields $8N+32=9N+36-162$, so $N=\mathbf{158}$ seeds.