BEST STUDENT EXAM OPEN

Texas A&M High School Math Contest

November 4, 2023

**Directions:** Answers should be simplified, and if units are involved include them in your answer.

**Problem 1.** What is the sum of the reciprocals of the solutions of the equation $n! + 3 = 3^{n-1}$?

**Problem 2.** Suppose that positive integers $x, y$ satisfy the equation $x^y + 1 = (x+1)^2$. What is the maximum possible value of $x^2 + y^2$?

**Problem 3.** How many prime numbers exist, which are less than 2023, and have a digit sum equaling 2?

**Problem 4.** We possess 5 white marbles and 10 black marbles. How many arrangements can we create when we place them in a sequence from left to right, ensuring that there is at least one black marble positioned immediately after every [1] white one?

**Problem 5.** How many solutions $(a, b, c)$ does the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = d$ have? Assume that $a, b, c, d$ are positive integers and $a < b < c$.

**Problem 6.** Let $a = 10^{2\times 2023} - 10^{2023} + 1$. What is $\frac{1}{2}(1 + \lfloor 2\sqrt{a} \rfloor)$? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

**Problem 7.** Evaluate $\lim_{n\to\infty} \left( \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{2n^2} \right)$.

**Problem 8.** Let $f(x) = \frac{4^x}{4^x+2}$. Evaluate the following sum

$$S = \sum_{k=1}^{2023} f\left(\frac{k}{2024}\right).$$

**Problem 9.** What is the largest possible number of elements in a subset $A$ of positive integers, where the sum of any three distinct elements in $A$ results in a prime number?

**Problem 10.** In triangle $\Delta ABC$, where $\angle A = 45^\circ$, point $D$ is located on line $BA$ such that $BD$ extends beyond point $A$, and $BD$ is equal in length to the sum of $BA$ and $AC$. Furthermore, we have two additional points, $K$ and $M$, positioned on line segments $AB$ and $BC$, respectively, such that the area of triangle $\Delta BDM$ matches the area of triangle $\Delta BCK$. What is the measure of angle $\angle BKM$?

**Problem 11.** How many functions $f$ from the set $1, 2, 3, 4$ to itself satisfy the condition that $f(f(x)) = f(x)$?

**Problem 12.** Let $\mathbb{N}$ represent the set of positive integers. Assume that a function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following properties

(a) $f(xy) = f(x) + f(y) - 1$, for all $x, y \in \mathbb{N}$.

(b) $f(x) = 1$ for finitely many $x$'s.

(c) $f(30) = 4$.

What is the value of $f(2)$?

**Problem 13.** For every positive integer $n$, let's define a set $A_n$ as follows:

$$A_n = \{x \in \mathbb{N} \mid \gcd(x, n) > 1\}.$$

We refer to a natural number $n > 1$ as a 'good' number if the set $A_n$ exhibits closure under addition. In other words, for any two numbers $x$ and $y$ in $A_n$, their sum $x + y$ also belongs to $A_n$. How many 'good' even numbers, not exceeding 2023, exist?

**Problem 14.** How many 3-digit prime numbers can be represented as $\overline{abc}$ where $b^2 - 4ac = 9$?

**Problem 15.** We choose a subset $S$ from the set $A = \{1, 2, 3, \ldots, 1001\}$ with the condition that for any two elements $x$ and $y$ in $S$, their sum $x + y$ is not in $S$. What is the largest possible size of the set $S$?

[1] In the version given during the exam it was written "a" instead of "every" which can be interpreted differently from what was originally meant; both interpretations were taken into account during the grading.

**Problem 16.** Suppose we have a triangle $\Delta ABC$ with side lengths $AB = 4$, $AC = 5$, and $BC = 6$. Let $A'$, $B'$, and $C'$ be the feet of the altitudes corresponding to the vertices $A$, $B$, and $C$, respectively. Furthermore, let $A''$, $B''$, and $C''$ be the points of intersection of the lines $AA'$, $BB'$, and $CC'$ with the circumcircle of the triangle $\Delta ABC$. What is the sum $\frac{AA''}{AA'} + \frac{BB''}{BB'} + \frac{CC''}{CC'}$?

**Problem 17.** Consider the function

$$g(x) = \left(x^2 + 7x - 47\right)\cosh(x),$$

where $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ . For every natural number $n$, $g^{(n)}(x)$ denotes the $n$th derivative of $g$. What is the sum of the numbers $n$ satisfying $g^{(n)}(0) = 2023$?

**Problem 18.** Evaluate the integral

$$\int_0^{\frac{\pi}{4}} (\cos^4 2x + \sin^4 2x)\ln(1 + \tan x)dx.$$

**Problem 19.** Let $a = \pi/2023$. Find the smallest positive integer $n$ such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer.

**Problem 20.** Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{22^2} + \frac{1}{23^2}}.$$