BEST STUDENT EXAM SOLUTIONS
Texas A&M High School Math Contest
November 4, 2023

Problem 1. What is the sum of the reciprocals of the solutions of the equation $n! + 3 = 3^{n-1}$?

Solution 1. It is easy to verify that the only solutions of the equation that are less than 6 are $n = 3$, and $n = 4$. For $n \geq 6$, the number $n!$ is divisible by 9, but $3^{n-1} - 3$ is not. Therefore, the answer is $\frac{1}{3} + \frac{1}{4} = \boxed{\dfrac{7}{12}}$. ∎

Problem 2. Suppose that positive integers x, y satisfy the equation $x^y + 1 = (x+1)^2$. What is the maximum possible value of $x^2 + y^2$?

Solution 2. The equation implies that $x^{y-1} = x + 2$, so $y = 1$ is impossible, hence $y \geq 2$ and $x|x^{y-1}$. In particular, $x|x + 2$, which implies $x|2$. Since $x = 1$ would lead to the contradiction 1=3, then we should try $x = 2$. The equation, therefore, reduces to $2^{y-1} = 2 + 2 = 4$, which has a unique solution $y = 3$, so the only possible value for $x^2 + y^2$ is $\boxed{13}$. ∎

Problem 3. How many prime numbers exist, which are less than 2023, and have a digit sum equaling 2?

Solution 3. Let N be such a prime number, so either $N = 2$ or N has two digits that are both 1, with all other digits being 0. Because N must be prime, the last digit must be 1, so

$$N = \underbrace{100\ldots00}_{n}1 = 10^n + 1 < 2023,$$

which implies that $n < 4$. Notice that $10^1 + 1 = 11$ and $10^2 + 1 = 101$ are prime numbers, but $10^3 + 1 = 1001$ is divisible by 11, so the answer is $\boxed{3}$. ∎

Problem 4. We possess 5 white marbles and 10 black marbles. How many arrangements can we create when we place them in a sequence from left to right, ensuring that there is at least one black marble positioned immediately after every [1] white one?

Solution 4. Rearrange the marbles by attaching to each white marble a black marble positioned immediately after it. This way, we have a total of 5 black marbles B and 5 "white-black" marbles WB to arrange in a row, from left to right. It is important to note that each arrangement of these 10 marbles corresponds to exactly one of the arrangements requested in the problem, and vice versa. Therefore, the total number of such arrangements can be calculated as $\binom{10}{5} = \frac{10!}{5!5!}$, which simplifies to 252. Hence, the answer is $\boxed{252}$. [2]∎

Problem 5. How many solutions (a, b, c) does the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = d$ have? Assume that a, b, c, d are positive integers and $a < b < c$.

Solution 5. Let $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = d$. Since a, b, c, d are positive integers and $a < b < c$, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} < 2,$$

so $d = 1$. We claim that $a = 2$. This is because $a \neq 1$, and if $a \geq 3$ we would have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} < 1,$$

which is impossible. So we must have

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$

[1] In the version given during the exam it was written "a" instead of "every" which can be interpreted differently from what was originally meant; both interpretations were taken into account during the grading.

[2] If at the end of the formulation of Problem 4 "every white one" is replace by "a white one" it can be very likely interpreted as follows: there exists a white marble with a black one positioned immediately after it. With this interpretation, there is only one arrangement which does not satisfies this condition: in this arrangement 10 black marbles are followed by 5 white ones. So in this interpretation the answer is $\binom{15}{5} - 1 = 3002$.

and therefore, $b \geq 3$. We claim that $b = 3$ because if $b \geq 4$ we would have

$$\frac{1}{2} = \frac{1}{b} + \frac{1}{c} \leq \frac{1}{4} + \frac{1}{5} = \frac{9}{20} < \frac{1}{2},$$

which is impossible. So $a = 2$, $b = 3$, hence $c = 6$. Therefore, the number of solutions is $\boxed{1}$. ■

Problem 6. Let $a = 10^{2\times 2023} - 10^{2023} + 1$. What is $\frac{1}{2}\left(1 + \lfloor 2\sqrt{a} \rfloor\right)$? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Solution 6. Let $b = 10^{2023}$, so that $a = b^2 - b + 1 = (b - \frac{1}{2})^2 + \frac{3}{4}$. In particular, we have

$$(b - \frac{1}{2})^2 < a < b^2,$$

which implies that

$$2b - 1 < 2\sqrt{a} < 2b,$$

hence $\lfloor 2\sqrt{a} \rfloor = 2b - 1$ and $\frac{1}{2}\left(1 + \lfloor 2\sqrt{a} \rfloor\right) = b = \boxed{10^{2023}}$. ■

Problem 7. Evaluate $\lim_{n\to\infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{2n^2} \right)$.

Solution 7. We have

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{2n^2} \right) &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{n}{1 + (\frac{k}{n})^2} \\
&= \int_0^1 \frac{dx}{1 + x^2} = \arctan x \,\Big|_0^1 \\
&= \boxed{\frac{\pi}{4}}. \blacksquare
\end{aligned}$$

Problem 8. Let $f(x) = \frac{4^x}{4^x + 2}$. Evaluate the following sum

$$S = \sum_{k=1}^{2023} f\left(\frac{k}{2024}\right).$$

Solution 8. We first observe that

$$f(1 - x) = \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{2 + 4^x},$$

hence

$$f(1 - x) + f(x) = \frac{4^x}{4^x + 2} + \frac{2}{2 + 4^x} = 1.$$

We can write the sum as

$$S = \left[f\left(\frac{1}{2024}\right) + f\left(\frac{2023}{2024}\right) \right] + \left[f\left(\frac{2}{2024}\right) + f\left(\frac{2022}{2024}\right) \right] + \cdots + \left[f\left(\frac{1011}{2024}\right) + f\left(\frac{1013}{2024}\right) \right] + f\left(\frac{1012}{2024}\right),$$

which is simplified to

$$S = 1011 + f\left(\frac{1}{2}\right) = 1011 + \frac{4^{\frac{1}{2}}}{4^{\frac{1}{2}} + 2} = \boxed{1011.5}. \blacksquare$$

Problem 9. What is the largest possible number of elements in a subset A of positive integers, where the sum of any three distinct elements in A results in a prime number?

Solution 9. The residues obtained when dividing the elements of A by 3 cannot encompass all three options, namely 0, 1, and 2:

$$3a + (3b + 1) + (3c + 2) = 3(a + b + c + 1) \geq 6$$

meaning that the sum of the three representatives with different residues is not a prime number, in particular. This implies that $|A| \leq 4$; otherwise, by the Pigeonhole Principle, three distinct elements of A would have the same residue modulo 3, and their sum would be divisible by 3 (and not less than 1+4+7=12). On the other hand, the example $A = \{1, 3, 7, 9\}$ shows that $|A| = 4$ is possible, therefore, the answer is $\boxed{4}$. ■

Problem 10. In triangle ΔABC, where $\angle A = 45°$, point D is located on line BA such that BD extends beyond point A, and BD is equal in length to the sum of BA and AC. Furthermore, we have two additional points, K and M, positioned on line segments AB and BC, respectively, such that the area of triangle ΔBDM matches the area of triangle ΔBCK. What is the measure of angle $\angle BKM$?

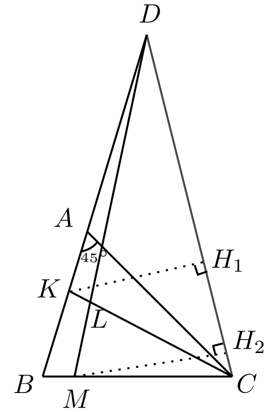
Solution 10. We have $AD = AC$, so $\angle ACD = \angle ADC = \frac{1}{2}(45°) = 22.5°$. Let L be the intersection of CK and DM. Denoting the area of a shape Δ by $[\Delta]$ we can write

$$[MLC] = [KLD]$$

due to the assumption that $[BDM] = [BCK]$. Therefore, if H_1, H_2 are the feet of the altitudes drawn from K, M, respectively, on the line CD, then we have

$$CD \cdot KH_1 = CD \cdot MH_2$$

so $KH_1 = MH_2$, hence $KM \parallel CD$ and $\angle BKM = \angle ADC = \boxed{22.5°}$. ■



Problem 11. How many functions f from the set $1, 2, 3, 4$ to itself satisfy the condition that $f(f(x)) = f(x)$?

Solution 11. Let us denote the range of f by R_f. The condition is equivalent to $f(y) = y$, for all $y \in R_f$, which means f acts as the identity function on R_f. We consider the following three cases.

(a) $|R_f| = 1$. In this case the function is constant, so there are 4 choices for f.

(b) $|R_f| = 2$. In this case, there are $\binom{4}{2} = 6$ ways to choose a 2-element subset of A as the range of f. The remaining 2 elements in the complement of R_f can each be mapped to any of the 2 elements in R_f, resulting in $2^2 = 4$ possible functions from the complement of R_f to R_f. Therefore, there are $6 \times 4 = 24$ functions in this case.

(b) $|R_f| = 2$. In this case, there are $\binom{4}{3} = 4$ ways to choose a 3-element subset of A as the range of f. The single remaining element in the complement of R_f can be mapped to any of the 3 elements in R_f, resulting in $3^1 = 3$ possible functions from the complement of R_f to R_f. Therefore, there are $4 \times 3 = 12$ functions in this case.

(d) $|R_f| = 4$. There is only one function in this case: the identity function.

So the total number of options is $4 + 24 + 12 + 1 = \boxed{41}$. ■

Problem 12. Let $\mathbb{N}$ represent the set of positive integers. Assume that a function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following properties

(a) $f(xy) = f(x) + f(y) - 1$, for all $x, y \in \mathbb{N}$.

(b) $f(x) = 1$ for finitely many x's.

(c) $f(30) = 4$.

What is the value of $f(2)$?

Solution 12. We claim that if $x \geq 2$ then $f(x) \geq 2$. If $f(x) = 1$, then

$$f(x^2) = f(x) + f(x) - 1 = 1,$$

and repeating this implies

$$f(x) = f(x^2) = f(x^4) = f(x^8) = \cdots = 1,$$

which contradict the condition (b). On the other hand, we can write

$$4 = f(30) = f(15) + f(2) - 1 = f(5) + f(3) - 1 + f(2) - 1,$$

hence $f(5) + f(3) + f(2) = 6$, and since $f(5), f(3), f(2) \geq 2$, we must have $f(2) = f(3) = f(5) = 2$, so the answer is $\boxed{2}$. ■

Problem 13. For every positive integer n, let's define a set A_n as follows:

$$A_n = \{x \in \mathbb{N} \mid \gcd(x, n) > 1\}.$$

We refer to a positive integer $n > 1$ as a 'good' number if the set A_n exhibits closure under addition. In other words, for any two numbers x and y in A_n, their sum $x + y$ also belongs to A_n. How many 'good' even numbers, not exceeding 2023, exist?

Solution 13. We assert that each 'good' even number must be a power of 2. Let n be a 'good' and even number, which means $n = 2^k m$, where $k \geq 1$ and m is an odd number. If $m > 1$, then $\gcd(m, n) = m > 1$, indicating that m is in the set A_n. Similarly, $\gcd(2^k, n) = 2^k > 1$, so 2^k is also in A_n. However,

$$\gcd(2^k + m, n) = \gcd(\underbrace{2^k + m}_{\text{odd}}, 2^k m) = \gcd(2^k + m, m) = \gcd(2^k, m) = 1,$$

meaning that $m + 2^k \notin A_n$, contradicting the assumption that n is a 'good' number. Therefore, we must have $m = 1$.

On the other hand, any integral power of 2 is a 'good' number: if $n = 2^k$, where $k \geq 1$, then A_n consists of all even numbers, which is closed under addition. Clearly, the inequality $2^k \leq 2023$ has 10 solutions $k = 1, 2, \dots, 10$, so the answer is $\boxed{10}$. ■

Problem 14. How many 3-digit prime numbers can be represented as $\overline{abc}$ where $b^2 - 4ac = 9$?

Solution 14. The condition $b^2 - 4ac = 9$ implies that the polynomial $P(x) := ax^2 + bx + c$ has rational roots, so it can be factored over rational numbers. In particular, we can write

$$kP(x) = (px + q)(rx + s),$$

for some integers p, q, r, s, and $k > 0$, such that $\gcd(p, q) = \gcd(r, s) = 1$. In particular, we have

$$pr = ka, \quad ps + qr = kb, \quad qs = kc$$

We claim that $k = 1$, and this is a particular case of the general Gauss' lemma, indeed. If k has a prime factor ν, then we have

$$\nu \mid pr, \quad \nu \mid ps + qr, \quad \nu \mid qs$$

implying that either $\nu \mid p$ or $\nu \mid r$, and, either $\nu \mid q$ or $\nu \mid s$. But since $\gcd(p, q) = \gcd(r, s) = 1$, we can only have two cases: either $\nu \mid p$ and $\nu \mid s$, or, $\nu \mid q$ and $\nu \mid r$. The first case, combined with the condition $\nu \mid ps + qr$, would result in $\nu \mid qr$, while the second case would result in $\nu \mid ps$, both of which contradict $\gcd(p, q) = \gcd(r, s) = 1$ due to the fact that ν is a prime. Therefore, k does not have any prime factor, so $k = 1$. In particular, we have

$$P(x) = (px + q)(rx + s),$$

hence

$$\overline{abc} = P(10) = (10p + r)(10r + s),$$

and this factorization is nontrivial since none of the factors $10p + r$ and $10q + s$ can be equal to one (indeed, each of them is greater than 10). Therefore, $\overline{abc}$ has to be a composite number, so the answer is $\boxed{0}$. ■

Problem 15. We choose a subset S from the set $A = \{1, 2, 3, \dots, 1001\}$ with the condition that for any two elements x and y in S, their sum $x + y$ is not in S. What is the largest possible size of the set S?

Solution 15. First, we observe that the subset $S = \{501, 502, \dots, 1001\}$, which contains 501 elements, satisfies the given condition.

Now, let's consider the case where $|S| > 501$. If t is the greatest element in the set S, then we can form the following $\lfloor \frac{t}{2} \rfloor$ pairs of (x, y):

$$(1, t-1), (2, t-2), \ldots, \left(\lfloor \frac{t}{2} \rfloor, \lceil \frac{t}{2} \rceil\right).$$

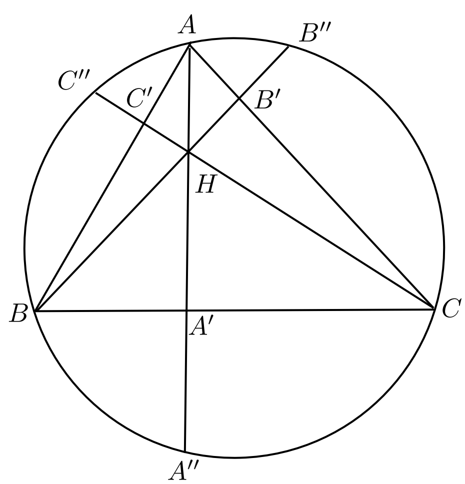
Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x \rceil$ denotes the least integer greater than or equal to x. The pairs listed above satisfy the condition that $x + y = t$, and since t is in S, at most one component of each pair can also be in S. Since all numbers $1, 2, \ldots, t-1$ have appeared as one of the components of these pairs, we should have $|S| \leq \lfloor \frac{t}{2} \rfloor$. But based on the assumption $|S| > 501$, we should have $\lfloor \frac{t}{2} \rfloor \geq 502$, hence $t \geq 1004$, which is a contradiction. Therefore, the answer is $\boxed{501}$.■

Problem 16. Suppose we have a triangle ΔABC with side lengths $AB = 4$, $AC = 5$, and $BC = 6$. Let A', B', and C' be the feet of the altitudes corresponding to the vertices A, B, and C, respectively. Furthermore, let A'', B'', and C'' be the points of intersection of the lines AA', BB', and CC' with the circumcircle of the triangle ΔABC. What is the sum $\frac{AA''}{AA'} + \frac{BB''}{BB'} + \frac{CC''}{CC'}$?

Solution 16. Let H be the orthocenter of ΔABC. It is well known that

$$HA' = A'A'', \quad HB' = B'B'', \quad HC' = C'C''.$$

Indeed, the angles $\angle BA''A$ and $\angle BCA$ are equal since they are subtended by the same arc. Then, by analyzing the corresponding right triangles, we can conclude that the angles $\angle BCB'$ and BHA' are equal. Since the former angle is equal to $\angle BCA$, and the latter angle is equal to $\angle BHA''$, we can deduce that $\angle BA''H = \angle BHA''$, which implies that $HA' = A'A''$.



We can then write

$$\frac{AA''}{AA'} = 1 + \frac{A'A''}{AA'} = 1 + \frac{HA'}{AA'} = 1 + \frac{HA' \cdot BC}{AA' \cdot BC} = 1 + \frac{[BHC]}{[ABC]},$$

where $[\Delta]$ denotes the area of the the shape Δ. After finding similar expressions for $\frac{BB"}{BB'}$ and $\frac{CC''}{CC'}$ we can write

$$\frac{AA''}{AA'} + \frac{BB''}{BB'} + \frac{CC''}{CC'} = 3 + \frac{[BHC] + [AHC] + [AHB]}{[ABC]} = 3 + \frac{[ABC]}{[ABC]} = 4,$$

so the answer is $\boxed{4}$. ■

Problem 17. Consider the function

$$g(x) = \left(x^2 + 7x - 47\right) \cosh(x),$$

where $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$. For every natural number n, $g^{(n)}(x)$ denotes the nth derivative of g. What is the sum of the numbers n satisfying $g^{(n)}(0) = 2023$?

Solution 17. Let $P(x) = x^2 + 7x - 47$, so we know that

$$P(0) = -47, \quad P'(0) = 7, \quad P''(0) = 2, \quad P^{(m)}(0) = 0 \quad \text{for } m \geq 3.$$

On the other hand, it is straightforward to observe that $\cosh^{(m)}(0) = 1$ for even values of m, and $\cosh^{(m)}(0) = 0$ for odd values of m. Applying the generalized product rule of differentiation, we have

$$g^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} P^{(k)}(0) \cosh^{(n-k)}(0),$$

which results in

$$g^{(n)}(0) = \begin{cases} -47 + n(n-1) & \text{if } n \text{ is even} \\ 7n & \text{if } n \text{ is odd} \end{cases}$$

Upon checking, the only odd solution of the equation $g^{(n)}(0) = 2023$ is $n = 289$, whereas the only even solution is $n = 46$, so the answer is $289 + 46 = \boxed{335}$.■

Problem 18. Evaluate the integral

$$\int_0^{\frac{\pi}{4}} (\cos^4 2x + \sin^4 2x)\ln(1 + \tan x)dx.$$

Solution 18. By making the substitution $x = \dfrac{\pi}{4} - u$ we get

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} (\cos^4 2u + \sin^4 2u)\ln\left(1 + \frac{1 - \tan u}{1 + \tan u}\right) du \\
&= \int_0^{\frac{\pi}{4}} (\cos^4 2u + \sin^4 2u)\ln\left(\frac{2}{1 + \tan u}\right) du = \ln 2 \int_0^{\frac{\pi}{4}} (\cos^4 2u + \sin^4 2u)du - I.
\end{aligned}$$

We solve for I and we obtain

$$\begin{aligned}
I &= \frac{\ln 2}{2} \int_0^{\frac{\pi}{4}} (\cos^4 2u + \sin^4 2u)du = \frac{\ln 2}{2} \int_0^{\frac{\pi}{4}} \left(1 - \frac{\sin^2 4u}{2}\right) du \\
&= \frac{\ln 2}{2} \int_0^{\frac{\pi}{4}} \left(\frac{3}{4} + \frac{\cos 8u}{4}\right) du = \boxed{\frac{3\pi \ln 2}{32}}.■.
\end{aligned}$$

Problem 19. Let $a = \pi/2023$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer.

Solution 19. By the product-to-sum identities, we have that $2\cos a \sin b = \sin(a+b) - \sin(a-b)$. Therefore, this reduces to a telescope series:

$$\begin{aligned}
\sum_{k=1}^{n} 2\cos(k^2 a)\sin(ka) &= \sum_{k=1}^{n} [\sin(k(k+1)a) - \sin((k-1)ka)] \\
&= -\sin(0) + \sin(2a) - \sin(2a) + \sin(6a) - \cdots - \sin((n-1)na) + \sin(n(n+1)a) \\
&= -\sin(0) + \sin(n(n+1)a) = \sin(n(n+1)a)
\end{aligned}$$

Thus, we need $\sin\left(\frac{n(n+1)\pi}{2023}\right)$ to be an integer; this can be only $\{-1, 0, 1\}$, which occur when $\frac{n(n+1)\pi}{2023}$ is an integer multiple of $\frac{\pi}{2}$, in other words, when $\frac{2n(n+1)}{2023}$ is an integer. Thus $2023 = 7 \cdot 17^2 | 2n(n+1)$. In particular, since $\gcd(n, n+1) = 1$, we must have exactly one of the following two cases: (1) $289 = 17^2 | n$, or (2) $289 = 17^2 | n+1$. Let us consider these cases separately.

Case (1): n is a multiple of 289, and either $7|n$ or $7|n+1$. In the former case, the smallest positive multiple of 289 that is divisible by 7 is 2023. In the latter case, we can write

$$n = 289k \equiv 2k \pmod{7},$$

therefore, we have

$$n + 1 \equiv 2k + 1 \equiv 0 \pmod{7},$$

implying that the smallest positive value of k is 3, hence $n = 3 \times 289 = 867$.

Case (2): $n + 1$ is a multiple of 289, and either $7|n$ or $7|n+1$. In the latter case, since the smallest positive multiple of 289 that is divisible by 7 is 2023, then $n = 2022$. In the former case, we can write

$$n + 1 = 289k \equiv 2k \pmod{7},$$

therefore, we have

$$n \equiv 2k - 1 \equiv 0 \pmod{7},$$

implying that the smallest positive value of k is 4, hence $n = 4 \times 289 - 1 = 1155$.

We conclude that the smallest value of n with the desired condition is $\boxed{867}$. ■

Problem 20. Determine the value of

$$S=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{22^2}+\frac{1}{23^2}}.$$

Solution 20. We have

$$\begin{aligned}
S &= \sum_{k=1}^{22}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} \\
&= \sum_{k=1}^{22}\sqrt{\frac{k^2(k+1)^2+(k+1)^2+k^2}{k^2(k+1)^2}} \\
&= \sum_{k=1}^{22}\sqrt{\frac{k^2(k+1)^2+2k(k+1)+1}{k^2(k+1)^2}} \\
&= \sum_{k=1}^{22}\sqrt{\frac{(k(k+1)+1)^2}{k^2(k+1)^2}} \\
&= \sum_{k=1}^{22}\frac{k(k+1)+1}{k(k+1)} \\
&= \sum_{k=1}^{22}\left(1+\frac{1}{k}-\frac{1}{k+1}\right) \\
&= 22+\left(1-\frac{1}{23}\right) \\
&= \boxed{22\frac{22}{23}=\frac{528}{23}}.
\end{aligned}$$ ■