DE Exam
Texas A&M High School Math Contest
November, 2023

1. Last year John received an inheritance. He payed 20% in federal taxes on the inheritance, and payed 10% of what he had left in state taxes. He payed a total of $10,500 for both taxes. How many dollars was the inheritance?

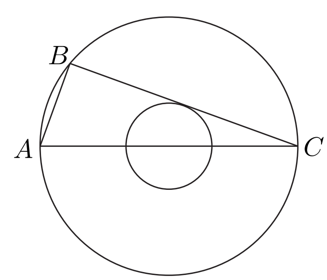
2. The first term of a sequence is 2023. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2023rd term of the sequence?

3. Two positive numbers are such that their difference, sum, and product are to one another as $1 : 7 : 24$. Find their sum.

4. The difference of the sum of legs of a right triangle and its hypothenuse is 8 inches. The height of the triangle drawn to the hypotenuse is 9.6 inches. Find the length of the longer leg.

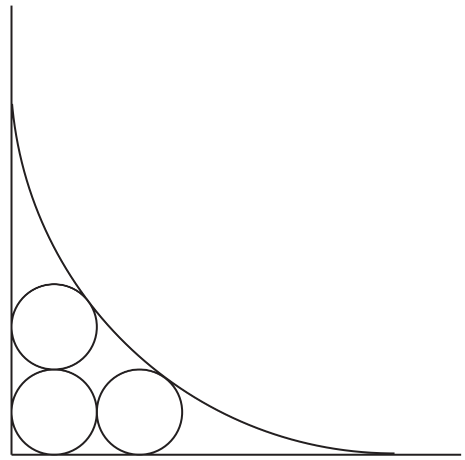
5. A regular polygon with n sides is inscribed in a circle of radius R. What is n, if the area of the polygon is $3R^2$?

6. The ratio of the radii of two concentric circles is 1:3. Let $\overline{AC}$ be a diameter of the larger circle, let $\overline{BC}$ be a chord of the larger circle that is tangent to the smaller circle. Find the radius of the larger circle if $AB = 12$.



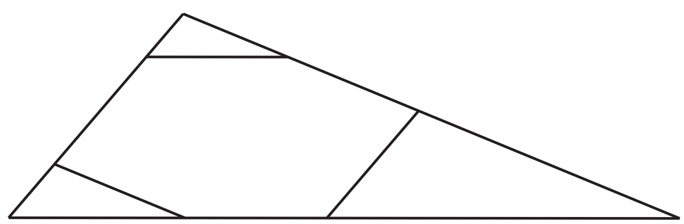
7. Find all roots of the equation $\sqrt{1+\sqrt{1+x}} = x$.

8. Three circles of radius s are drawn inside a right angle. The first circle is tangent to both sides of the angle, the second is tangent to the first circle and one side of the angle, and the third is tangent to the first circle and the other side of the angle. A circle of radius $r > s$ is tangent to both sides of the angle and to the second and the third circles. Find r/s.



9. Trapezoid is divided by its diagonals into four parts. What is the area of the trapezoid if the areas of the parts adjacent to the bases are 16 and 25?

10. Three lines parallel to the sides of a triangle cut from it three smaller triangles, so that a hexagon remains. If all sides of the hexagon are of the same length, and the lengths of the sides of the triangle are a, b, c, then what is the length of a side of the hexagon?



11. Six distinct integers are picked at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?

12. A sphere is inscribed in a tetrahedron $ABCD$. The four planes tangent to the sphere parallel to (but distinct from) the planes of the four faces of $ABCD$ cut off smaller tetrahedra. Let F_{ABCD} be the sum of length of the edges of the four smaller tetrahedra divided by the sum of the lengths of the edges of $ABCD$. What is the largest possible value of F_{ABCD} among all tetrahedra $ABCD$.

13. Five points are chosen inside or on a square of side 1. Find the smallest possible a such that it is always possible to select one pair of points from these five such that the distance between them is equal to or less than a.

14. Determine the number of real solutions of the equation

$$\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{3} \right\rfloor + \left\lfloor \frac{a}{5} \right\rfloor = a,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

15. Let $f(x)$ and $g(x)$ be different quadratic polynomials of the form x^2+px+q such that $f(1)+f(10)+f(100)=g(1)+g(10)+g(100)$. Find all x such that $f(x)=g(x)$.

16. The function $F(x)$ is defined for all real numbers except 0 and 1, and satisfies

$$F(x)+F\left(\frac{x-1}{x}\right)=1+x$$

for all $x\neq 0,1$. Find $F(5)$.

17. Find the largest prime p for which p^2-2, $2p^2-1$, and $3p^2+4$ are prime numbers.

18. How many pairs (x,y) of positive integers exist such that $\sqrt{x}+\sqrt{y}=\sqrt{2023}$?

19. Let each of the numbers $x_1,x_2,\ldots,x_{2023}$ be independently equal to -1, 0, or 1. What is the smallest possible value of the sum of their pairwise products $\sum_{1\leq i<j\leq 2023} x_ix_j$?

20. Call a set of integers *sparse* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1,2,\ldots,12\}$, including the empty set, are sparse?