# Effective Non-Vanishing of Class Group $L$-Functions for Biquadratic CM Fields

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# Statement of Results

**Theorem (B-S, Peirce, W)**

*Let* $d_1 > 0$ *and* $d_2 < 0$ *be square-free, co-prime integers with* $d_1 \equiv 1 \mod 4$ *and* $d_2 \equiv 2$ *or* $3 \mod 4$. *Assume* $K = \mathbb{Q}(\sqrt{d_1})$ *has class number one and let* $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. *Then if*

$$|d_2| \geq C_1(d_1) := (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1) + 2)\right\},$$

*then there exists a character* $\chi \in \widehat{Cl(\mathcal{O}_E)}$ *such that* $L\left(\chi, \frac{1}{2}\right) \neq 0$.

# Connection to Eisenstein Series

Under our assumptions on $K$ and $E$, the average formula becomes

$$\frac{1}{h_E} \sum_{\chi \in \widehat{Cl(\mathcal{O}_E)}} L(\chi, \tfrac{1}{2}) = \left( \frac{2^n d_1}{\sqrt{|d_2|}} \right)^{\frac{1}{2}} \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_K^\times]} E_K(z_{\mathcal{O}_E}, \tfrac{1}{2})$$

where the special point is

$$z_{\mathcal{O}_E} = \left( \sqrt{d_2}, \sqrt{d_2} \right) \in \mathbb{H}^2.$$

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where the special point is

$$z_{\mathcal{O}_E} = \left(\sqrt{d_2}, \sqrt{d_2}\right) \in \mathbb{H}^2.$$

From this formula, it suffices to show that

$$E_K(z_{\mathcal{O}_E},\tfrac{1}{2}) \neq 0.$$

# Decomposition of the Eisenstein Series

**Proposition**

We have

$$E_K(z_{\mathcal{O}_E}, \tfrac{1}{2}) = M(d_1, d_2) + H(d_1, d_2)$$

where

$$M(d_1, d_2) = \sqrt{|d_2|}\left[\frac{2R_K}{\sqrt{d_1}}\left(\log(|d_2|) - \log\left(\frac{\pi^2}{d_1}\right) - 2(\gamma_{\mathbb{Q}} + \log(4))\right) + 2\gamma_K\right]$$

and

$$H(d_1, d_2) = \sqrt{|d_2|}\sum_{\gamma\in\mathcal{O}_K}\sum_{0\neq\nu\in\mathcal{O}_K^\vee} c_\nu(\gamma y(z_{\mathcal{O}_E}))e^{2\pi i Tr(\gamma\nu x)}.$$

# The plan of the proof

- By the previous proposition and the reverse triangle inequality,

$$|E_K(z_{\mathcal{O}_{d_2}}, \tfrac{1}{2})| \geq |M(d_1, d_2)| - |H(d_1, d_2)|.$$

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- Hence it suffices to show $|M(d_1, d_2)| > |H(d_1, d_2)|$.
- We will give an upper bound for $|H(d_1, d_2)|$ and a lower bound for $|M(d_1, d_2)|$.

# An Upper Bound for $|H(d_1, d_2)|$

**Proposition**

If

$$|d_2| \geq (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1) + 2)\right\},$$

then

$$|H(d_1, d_2)| \leq 6.80 \times 10^{-401}.$$

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The proof involves a very complicated argument to effectivize an upper bound of Bauer.

# A Lower Bound for $M(d_1, d_2)$

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The proof uses Ihara's lower bound

$$\gamma_K > -2(\log(4d_1) + 2)(\log(\sqrt{d_1}) - \gamma_{\mathbb{Q}} - \log(4\pi))$$

and the lower bound
$$R_K > \log(2\sqrt{d_1}).$$

# Summary

- For
$$|d_2| \geq (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1)+2)\right\},$$
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- Thus $|M(d_1,d_2)| > |H(d_1,d_2)|$, implying $|E_K(z_{\mathcal{O}_E},\frac{1}{2})| > 0$.
- Hence, by the average formula, there exists a $\chi \in \widehat{Cl(\mathcal{O}_E)}$ such that $L(\chi,\frac{1}{2}) \neq 0$.

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- Hence for all
$$|d_2| \geq 2.77028 \times 10^{13},$$
there exists a $\chi$ such that $L(\chi, \frac{1}{2}) \neq 0$.

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- The restrictions on $d_1$ and $d_2$ were made to simplify the presentation.
- A version of the main result holds for *any* CM extension $E$ of $K$ when $K$ has class number one.
- One has reduced the question of the existence of non-vanishing $L(\chi, \frac{1}{2})$ to a (large!) finite calculation.