

Refining Fewnomial Theory for Certain 2×2 Systems

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Abstract

Finding the correct extension of Descartes' Rule to multivariate polynomial systems remains a difficult open problem. We focus on pairs of polynomials where the first has 3 terms and the second has m , and we say these systems are of type $(3, m)$. For these systems, recent work has determined that the maximum finite number of roots in the positive quadrant lies between $2m - 1$ and $\frac{2}{3}m^3 + 5m$. The techniques applied so far are variants of Rolle's Theorem and a forgotten result of Polya on the Wronskian of certain analytic functions. We add to these techniques by considering curvature, as a step toward establishing sharper upper bounds. We also build new extremal examples of minimal height.

1 Introduction

Descartes' Rule of Signs gives a simple method for determining the maximum number of positive roots in a univariate polynomial. Since finding real roots of systems of polynomials in many variables occurs frequently in applications, extending Descartes' Rule to multivariable systems has become an important, open problem. Currently, work on generalizations of Descartes' Rule to multivariable systems only yields loose bounds on the number of positive real roots.

Definitions. We define a $n \times m$ system as a system of n -polynomials and m -variables. Furthermore, we define systems where one is a trinomial and the other is a k -nomial as a **System of Type $(3, k)$** .

Let N_m be the maximal finite number of roots in \mathbb{R}_+^2 of a 2×2 system of type $(3, m)$. We have the following bounds on N_m (from the work of [LRW03, GNR07, KPT15]) :

$$2m - 1 \leq N_m \leq \min \left\{ 2^m - 2, \frac{2}{3}m^3 + 5m \right\}$$

Techniques applied to finding bounds on N_m have been variants of Rolle's Theorem and a forgotten result of Polya on the Wronskian.

Finding extremal examples of systems of minimal height also remains a challenging problem. For a system of type $(3, 3)$, it has been proven that $N_m = 5$ exactly. However, the first system of type $(3, 3)$ with 5 roots was only discovered in 2000. Furthermore, a system of type $(3, 4)$ with 7 roots was discovered in 2007. These examples give insight to fewnomial theory that can lead to proving new facts.

2 The Project

The first aspect of this project's focus was on building extremal examples of minimal height showing that there exists a family of 2×2 systems of type (3, 4) with 7 roots in \mathbb{R}_+^2 . We start by looking at the following system of type (3, 3) that is known to have 5 roots in \mathbb{R}_+^2 :

$$\begin{aligned} f(x_1, x_2) &:= x_1^5 - \frac{49}{95}x_1^3x_2 + x_2^6 \\ g(x_1, x_2) &:= x_2^5 - \frac{49}{95}x_1x_2^3 + x_1^6 \end{aligned} \tag{1}$$

First, we reduce System 1 by multiplying $f(x_1, x_2)$ by $\frac{1}{x_1^3x_2}$ and $g(x_1, x_2)$ by $\frac{1}{x_1x_2^3}$. We perform a change of variables on System 1 to get

$$\begin{aligned} r(u, v) &:= u - \frac{49}{95} + v \\ s(u, v) &:= u^{\frac{1}{7}}v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}}v^{\frac{-1}{7}} \end{aligned} \tag{2}$$

where $u = x_1^2x_2^{-1}$ and $v = x_1^{-3}x_2^5$. Solving for System 2, we get the following algebraic function:

$$G(u) := u^{\frac{1}{7}} \left(\frac{49}{95} - u \right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left(\frac{49}{95} - u \right)^{\frac{-1}{7}} = 0 \tag{3}$$

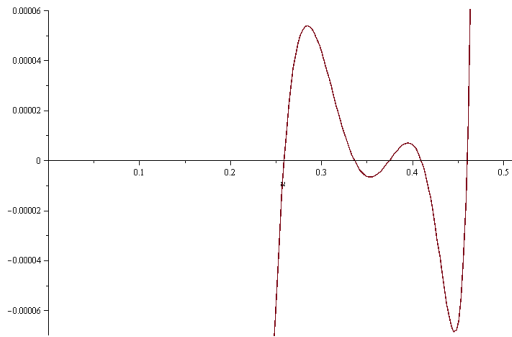


Figure 1: Graph of $G(u)$

We want to create a hump that would intersect $G(u)$ at 7 distinct locations. This hump is created using the monomial

$$H(u) := cu^a \left(\frac{49}{95} - u \right)^b \tag{4}$$

where we choose the parameters (a, b, c) . For some (a_0, b_0, c_0) that creates an $H(u)$ that intersects $G(u)$ 7 times, we will create a 2×2 system of type (3, 4) that has 7 roots in \mathbb{R}_+^2 .

3 Methods

We will only focus on finding the roots of $G(u)$ that lie in the interval $(0, \frac{49}{95})$. This is because only the roots that lie in this interval will result in a solution set $(u, v) \in \mathbb{R}_+^2$ of System 2. This will in turn insure that the original roots of System 1 lie in \mathbb{R}_+^2 . We do a change of variables and rescaling on System 2 to get

$$\begin{aligned} t(p, q) &:= p^7 - \frac{49}{95} + q^7 \\ z(p, q) &:= pq^4 - \frac{49}{95}q + p^{16} \end{aligned} \tag{5}$$

where $p = u^{\frac{1}{7}}$ and $q = v^{\frac{1}{7}}$.

Definition. Given any $d, e \in \mathbb{N}$ and $f, g \in \mathbb{C}[x]$ with $\deg(f) \leq d$ and $\deg(g) \leq e$, the **Sylvester Matrix** of (f, g) of format (d, e) is shown in Figure 2. The **Resultant** of f and g (denoted $\text{Res}_{(d,e)}(f, g)$) is the determinant of their Sylvester Matrix.

(Without loss of generality, assume $f(x) = a_0 + \dots + a_d x^d$ and $g(x) = b_0 + \dots + b_e x^e$.)

$$\text{SYL}_{(d,e)}(f, g) = \begin{pmatrix} a_0 & a_1 & \dots & a_d & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_d & \dots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & \\ 0 & \dots & 0 & a_0 & a_1 & \dots & a_d \\ b_0 & b_1 & \dots & b_e & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_e & \dots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & \\ 0 & \dots & 0 & b_0 & b_1 & \dots & b_e \end{pmatrix}$$

Figure 2: Sylvester Matrix of (f, g) of format (d, e)

We note that the matrix is a square $(e + d) \times (d + e)$ matrix.

We then get the $\text{Res}_{(7,4)}(t, z)$ in terms of q by treating the coefficients as functions of p . We back substitute $u = p^7$ from the resultant and evaluate the zero set between the interval $(0, \frac{49}{95})$. We note that there are 5 solutions in the interval $(0, \frac{49}{95})$. Using the derivatives of system 5 and the resultant, we were able to find where each local maximum and minimum occurs in $G(u)$ in the interval $(0, \frac{49}{95})$.

Now we look at finding parameters (a, b, c) that would lead to 7 roots to $G(u) = H(u)$. The graph of $H(u)$ is a horizontal line with a hump. We want this hump's points of inflection to be between at the endpoint of some interval (i_1, i_2) . We want the peak to be at the midpoint of this interval, $(\frac{i_1+i_2}{2})$. By taking derivatives of $H(u)$ and with some algebra we get the following functions for a, b , and c based on the midpoint, $m = \frac{i_1+i_2}{2}$, the radius, $d = \frac{i_2-i_1}{2}$, and the desired height of the hump, h :

$$a = \frac{m^2}{d^2} \left(1 - \frac{95m}{49} \right) + \frac{95m}{49} \tag{6}$$

$$b = \frac{49a}{95m} - a \quad (7)$$

$$c = h \cdot \left(\frac{a+b}{49/95} \right)^{a+b} \cdot \frac{1}{a^a b^b} \quad (8)$$

We set $G(u) = H(u)$, and determine if there are 7 intersections. When we have a success, we define

$$G_2(u) := G(u) - H(u).$$

So, $G_2(u)$ should have 7 roots in $(0, \frac{49}{95})$. We undo the substitution we did before, and undo the change of variable to acquire a 2×2 system of type (3, 4) with 7 roots in \mathbb{R}_+^2 .

4 Results

We found the roots of $G(u)$ in order to give a total of six regions to construct $H(u)$ around to ensure 7 intersections between $H(u)$ and $G(u)$: 4 between consecutive each root, 1 between 0 and the first root, and 1 between the last root and $\frac{49}{95}$. For each region, we found parameters (a, b, c) such that the peak would be at the center of these roots. Additionally, we found parameters (a, b, c) such that the peak of $H(u)$ is aligned the peaks of $G(u)$. After this initial analysis, we started looking at humps that would peak closer to the endpoints of each region. We were able to find multiple systems for each region. Figure 3 shows a visual representation of these 6 regions.

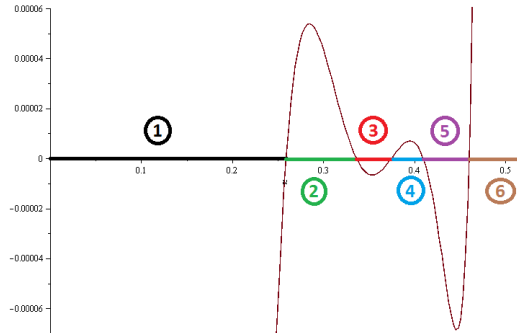


Figure 3: $G(u)$ is red and $H(u)$ is blue

For example, Figure 4 will show that the function

$$H(u) := 7.290836369 \cdot 10^{154} u^{158} \left(\frac{49}{95} - u \right)^{115}$$

is a function that creates a hump between Region 2, the area between the first 2 roots. Figure 5 shows that $G_2(u)$ created for this function has a total of 7 roots.

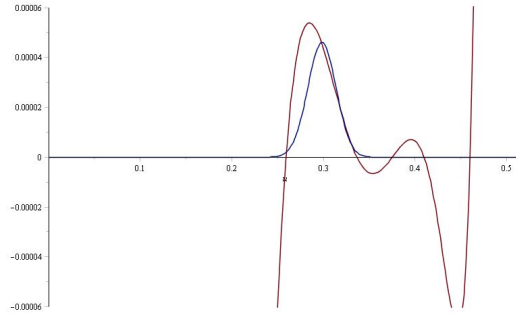


Figure 4: $G(u)$ is red and $H(u)$ is blue

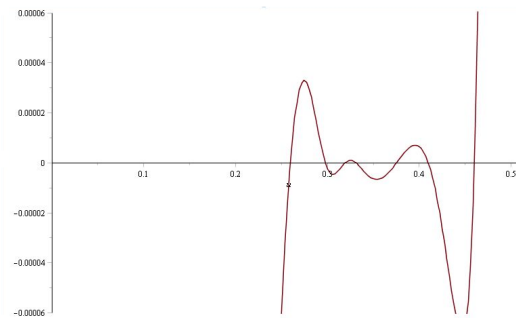


Figure 5: $G_2(u)$

With each $H(u)$ constructed, we plotted the points (a, b) to find areas on the graph where we can find more possible parameters of (a, b) that give 7 intersections with an appropriate value of c . We note that scalar multiples of (a, b) that give a 7 root system also give a 7 root system. Figure 6 plots these (a, b) along with a shaded area represented values of (a, b) that could possibly give a 7 root system. The areas are color coded to represent where the humps were inserted, according to the regions in Figure 3.

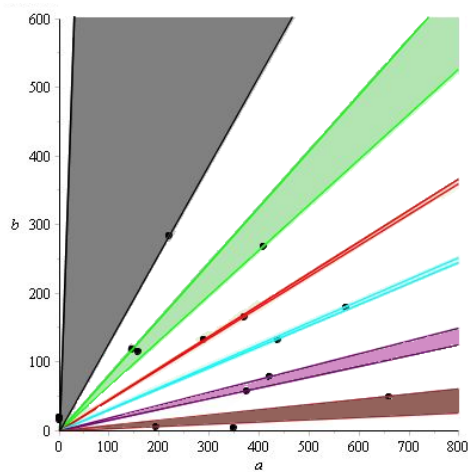


Figure 6: Areas with Possible Parameters (a, b)

The area between these lines are believed to contain more parameters that would give a system with 7 roots. Note that these are not bounds for finding parameters (a, b) that will give a 7 intersections when the hump is placed between the first 2 roots. Instead, this area between each colored line gives an area that can be studied to give additional systems with 7 roots in \mathbb{R}_+^2 .

From testing parameters for each region, we noticed that for all regions but the first region, the coefficient c would have to be very large (on the scale of over 10^{100}). However, once we started trying to find parameters (a, b) in Region 1 (between 0 and the first root), we were able to construct systems with smaller coefficients c . We were ultimately able to find the following $H(u)$ that has 7 intersections with $G(u)$.

$$H(u) = -5807u \left(\frac{49}{95} - u \right)^{12} \quad (9)$$

We were then able to construct a $G_2(u)$ with this polynomial and, ultimately, we got the following 2×2 system of type $(3, 4)$ with 7 roots in \mathbb{R}_+^2 .

$$\begin{aligned} f_2(x_1, x_2) &:= x_1^5 - \frac{49}{95}x_1^3x_2 + x_2^6 \\ g_2(x_1, x_2) &:= x_1^{33}x_2^5 - \frac{49}{95}x_1^{34}x_2^3 + x_1^{39} + 5807x_2^{62} \end{aligned} \quad (10)$$

We would like to note that other parameters (a, b, c) were found, and other systems were created. However, the system shown above is the simplest system.

5 Conclusion

Through this project, we were able to find a relatively simple example of a 2×2 system of type $(3, 4)$ with 7 roots in \mathbb{R}_+^2 . The construction of the humps created can be further studied to possibly find a more simple example. The results of this project can give insight on how polynomial systems work. Although exact bounds were not found for (a, b, c) that will give 7 roots, we were able to find areas that will contain parameters that allow for construction of new systems (as shown in figure 6).

A new direction for this project is to try to do the same methods provided in this paper to other 2×2 systems of type $(3, 3)$ to get new systems of type $(3, 4)$. One of this projects goals was to find tighter bounds on N_m , but was not worked on in the scope of this REU. A new direction would also be finding a tighter bound on systems of type $(3, 4)$ than that mentioned in the introduction. To do so, we will look how to prove the following conjecture.

Conjecture. Let us fix an integer k . Then the maximum number of roots of

$$\sum_{i=1}^k c_i u^{a_i} (1-u)^{b_i} \text{ in } (0, 1)$$

(over all real a_i, b_i , and c_i) is $O(k)$.

6 The REU Program

I felt that this REU experience has been a great exposure to the world of math research. I had never been exposed to research or algebraic geometry before this program. I felt that my mentor, Dr. Rojas, did a great job at introducing me to this new world by giving two weeks of lectures into the subject. The program also did a great job at familiarizing me to other fields through biweekly presentations to the whole REU. Additionally, I enjoyed how this program got me acquainted with different mathematical computer software, including Maple and Bertini. This program has solidified my desire to continue to do research and continue my education to graduate school.

References

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