Zeros of the Modular Form $E_k E_l - E_{k+l}$

Sarah Reitzes and Polina Vulakh

Tufts University, Bard College

pvulakh@gmail.com

July 23, 2015

Zeros of Modular Forms

Zeros of Modular Forms

- A complex-valued function f is a **modular form** if:

Zeros of Modular Forms

- A complex-valued function f is a **modular form** if:
  1. for all $z \in \mathbb{H} \cup \{\infty\}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ we have $f(\gamma(z)) = f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$. Then k is called the weight of f. (*Transformation Law*)

Zeros of Modular Forms

- A complex-valued function f is a **modular form** if:
  1. for all $z \in \mathbb{H} \cup \{\infty\}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ we have $f(\gamma(z)) = f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$. Then k is called the weight of f. (*Transformation Law*)
  2. f is holomorphic (complex differentiable) for every point $z \in \mathbb{H} \cup \{\infty\}$, so f can be expanded in a power series in z around any point $z_0 \in \mathbb{H}$

Zeros of Modular Forms

- A complex-valued function f is a **modular form** if:
  1. for all $z \in \mathbb{H} \cup \{\infty\}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ we have $f(\gamma(z)) = f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$. Then k is called the weight of f. (*Transformation Law*)
  2. f is holomorphic (complex differentiable) for every point $z \in \mathbb{H} \cup \{\infty\}$, so f can be expanded in a power series in z around any point $z_0 \in \mathbb{H}$

We look at modular forms of positive, even weight.

Zeros of Modular Forms

- A complex-valued function f is a **modular form** if:
  1. for all $z \in \mathbb{H} \cup \{\infty\}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ we have $f(\gamma(z)) = f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$. Then k is called the weight of f. (*Transformation Law*)
  2. f is holomorphic (complex differentiable) for every point $z \in \mathbb{H} \cup \{\infty\}$, so f can be expanded in a power series in z around any point $z_0 \in \mathbb{H}$

  We look at modular forms of positive, even weight.
- The **valence formula** tells us how many zeros f has.

$$\frac{k}{12} = \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{\substack{z \neq i,\rho \\ z \in \mathbb{H}}} v_z(f)$$

Zeros of the Eisenstein Series in $\mathcal{F}$

Eisenstein series of weight k:

$$E_k(z) = \frac{1}{2}\sum_{\substack{(c,d)=1 \\ c,d\in\mathbb{Z}}} \frac{1}{(cz+d)^k} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n z}$$

It has been proven that the zeros of the Eisenstein series lie on the arc of the fundamental domain $\mathcal{F} = \{z = x + iy \in \mathbb{H} : x \in (-\frac{1}{2}, \frac{1}{2}), |z| \geq 1\}$ (RSD, 1970).

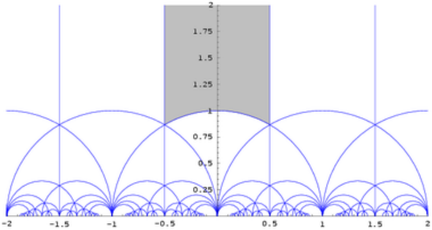


Figure: $\mathcal{F}$

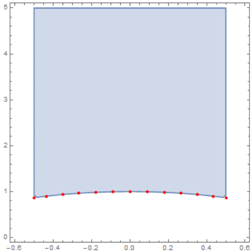


Figure: Zeros of E_{70}

Zeros of $E_k E_l - E_{k+l}$

Conjecture:

The zeros of $E_k E_l - E_{k+l}$, a modular form of weight $k + l$, lie on the boundary of $\mathcal{F}$.

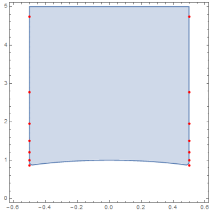


Figure: Zeros of $E_{50}^2 - E_{100}$

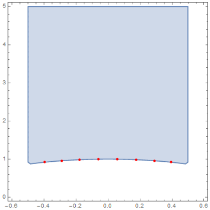
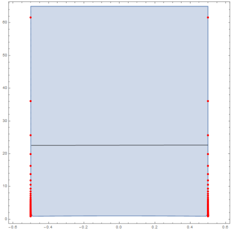


Figure: Zeros of $E_{60} E_8 - E_{68}$

Conjecture:

The zeros of $E_k^2 - E_{2k}$, a modular form of weight $2k$, lie on the lines $x = \pm\frac{1}{2}$ in $\mathcal{F}$.

Since $E_k(\frac{1}{2} + iy)$ is real-valued, we prove the desired number of zeros $(\lfloor \frac{k}{6} \rfloor - (1+n))$ via IVT using points of the form $\frac{1}{2} + iy_m$ where $y_m = \frac{\tan(\theta_m)}{2}$ for $\theta_m = \frac{m\pi}{k}$ where $m \in \mathbb{Z}$ such that $\lceil \frac{k}{3} \rceil \leq m < \frac{k}{2} - n$..
Why $-n$?
We run into problems for $y \geq \frac{c_0\sqrt{k}}{\sqrt{\log k}}$, so n is the number of zeros with y past this range.
However, there exists a method involving the Fourier expansion that proves the location of zeros for which $y > c_1\sqrt{k \log k}$, so we lose very few zeros altogether.



Write $E_k(\frac{1}{2}+iy) = M_k(\frac{1}{2}+iy) + R_k(\frac{1}{2}+iy)$ where M_k corresponds to $c^2+d^2 \leq 1$ - except for $(c,d)=(1,1)$ - and R_k corresponds to all other (c,d).
Then

$$\begin{aligned} E_k^2(\tfrac{1}{2}+iy) - E_{2k}(\tfrac{1}{2}+iy) &= (M_k(\tfrac{1}{2}+iy) + R_k(\tfrac{1}{2}+iy))^2 \\ &\quad - (M_{2k}(\tfrac{1}{2}+iy) + R_{2k}(\tfrac{1}{2}+iy)) \\ &= M_k(\tfrac{1}{2}+iy)^2 + 2M_k(\tfrac{1}{2}+iy)R_k(\tfrac{1}{2}+iy) \\ &\quad + R_k(\tfrac{1}{2}+iy)^2 - M_{2k}(\tfrac{1}{2}+iy) - R_{2k}(\tfrac{1}{2}+iy) \end{aligned}$$

We know $|R_k(\frac{1}{2}+iy)| < \frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}}$, which is decreasing in k, and since $M_k(\frac{1}{2}+iy) = 1 + \frac{1}{(\frac{1}{2}+iy)^k} + \frac{1}{(-\frac{1}{2}+iy)^k}$, we know $|M_k(\frac{1}{2}+iy)| \leq 3$. Then we want to show

$$|M_k(\tfrac{1}{2}+iy_m)^2 - M_{2k}(\tfrac{1}{2}+iy_m)| > 8\left(\frac{9+12y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}}\right)$$

Approximating $E_k^2 - E_{2k}$ (cont.)

For our points $\frac{1}{2} + iy_m$, we have a lower bound

$$|M_k(\tfrac{1}{2} + iy_m)^2 - M_{2k}(\tfrac{1}{2} + iy_m)| \geq \frac{4(\frac{1}{4} + y_m^2)^{\frac{k}{2}} - 2}{(\frac{1}{4} + y_m^2)^k}$$

so we want to show

$$\frac{4(\frac{1}{4} + y_m^2)^{\frac{k}{2}} - 2}{(\frac{1}{4} + y_m^2)^k} > 8\left(\frac{9+12y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}}\right)$$

For large y, this is not true: specifically for $y \geq c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$ where $c_0 \leq \frac{1}{\sqrt{8}}$, so we work with $y_m < c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$.

By simplifying further, we have $\left(\frac{\frac{9}{4}+y_m^2}{\frac{1}{4}+y_m^2}\right)^{\frac{k}{2}} > c_2 y_m$ where $c_2 = \frac{38}{\sqrt{3}} + 24$.

This is true for $k \geq c_2$, so we have proved $|M_k(\frac{1}{2} + iy_m)^2 - M_{2k}(\frac{1}{2} + iy_m)| > 8\left(\frac{9+12y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}}\right)$.

If we rewrite $\frac{1}{2}+iy_m = re^{i\theta_m}$, we have

$$M_k(re^{i\theta_m})^2 - M_{2k}(re^{i\theta_m}) = \frac{4r^k(-1)^m + 2}{r^{2k}}$$

For $\theta_m = \frac{m\pi}{k}$ where $m \in \mathbb{Z}$ such that $\lceil \frac{k}{3} \rceil \leq m < \frac{k}{2} - n$, this yields $\lfloor \frac{k}{6} \rfloor - n$ sign changes corresponding to $\lfloor \frac{k}{6} \rfloor - n - 1$ zeros by IVT.

Extending this to general $E_k E_l - E_{k+l}$

Recall that $B_{k,l}$ =number of zeros of $E_k E_l - E_{k+l}$ for which $x = \frac{1}{2}$.

Conjecture:

$(k \geq l)$ The number of zeros $E_k E_l - E_{k+l}$ for which $x = \frac{1}{2}$ is at least that of $E_l^2 - E_{2l}$. In other words, $B_{k,l} \geq B_{l,l}$.

k\l	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
10	0	0	1	0	0	1	0	1	1	0	1	1	0	1	1	1	1	1	1	1	1
12	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
16	0	1	1	1	1	2	1	1	2	1	1	2	1	2	2	1	2	2	1	2	2
18	0	1	1	1	2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
20	1	1	1	2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
22	0	1	1	1	2	2	2	2	3	2	2	3	2	2	3	2	2	3	2	3	3
24	1	1	1	1	2	2	2	3	2	3	3	2	3	3	3	3	3	3	3	3	3
26	1	1	1	2	2	2	3	2	3	3	3	3	3	3	3	3	3	3	3	3	3
28	0	1	1	1	2	2	2	3	3	3	3	4	3	3	4	3	3	4	3	3	4
30	1	1	1	1	2	2	2	3	3	3	4	3	4	4	3	4	4	4	4	4	4
32	1	1	1	2	2	2	3	2	3	4	3	4	4	4	4	4	4	4	4	4	4
34	0	1	1	1	2	2	2	3	3	3	4	4	4	4	5	4	4	5	4	4	5
36	1	1	1	2	2	2	2	3	3	3	4	4	4	5	4	5	5	4	5	5	5
38	1	1	1	2	2	2	3	3	3	4	3	4	5	4	5	5	5	5	5	5	5
40	1	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	5	6	5	5	6
42	1	1	1	2	2	2	2	3	3	3	4	4	4	5	5	5	6	5	6	6	5
44	1	1	1	2	2	2	3	3	3	4	4	4	5	4	5	6	5	6	6	6	6
46	1	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	6	7
48	1	1	1	2	2	2	3	3	3	3	4	4	4	5	5	5	6	6	6	7	7
50	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	5	6	7	7	7

Example: $E_k E_{34} - E_{k+34}$

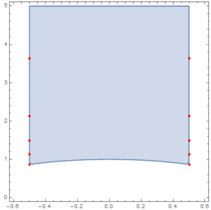


Figure: k=34

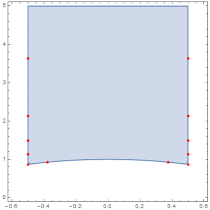


Figure: k=40

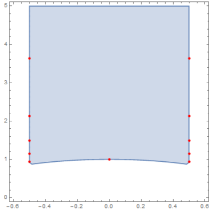


Figure: k=44

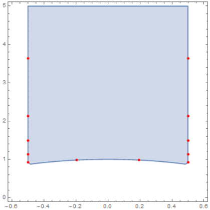


Figure: k=50

Our main term becomes

$$M_k(re^{i\theta})M_l(re^{i\theta}) - M_{k+l}(re^{i\theta}) = \frac{r^{2l+k}2\cos(\theta k)+r^{2k+l}2\cos(\theta l)+r^{k+l}2\cos(\theta(k-l))}{r^{2(k+l)}}$$

If we rewrite $k = l + d$ and let $\theta_m = \frac{m\pi}{l}$ for $\lceil \frac{l}{3} \rceil \leq m < \frac{l}{2}$,

$$\frac{r^{3l+d}2(-1)^m \cos(\frac{m\pi}{l}d) + r^{3l+2d}2(-1)^m + r^{2l+d}2\cos(\frac{m\pi}{l}d)}{r^{4l+2d}}$$

as our main term instead.

By splitting this up into three cases for $d \equiv 0, 2, 4 \pmod 6$, we follow a similar method to show that $E_k E_l - E_{k+l}$ has at least $\lfloor \frac{l}{6} \rfloor - n - 1$ zeros or which $x = \frac{1}{2}$.