A Faster Randomized Algorithm for Counting Roots in $\mathbb{Z}/(p^k)$

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Outline

Today we will...

- Review the randomized algorithm for counting mod p^k
- See some specific examples regarding computational time
- Go over the complexity bound of the algorithm
- Discuss a bound on the number of roots mod p^k

Factorization

- Counting roots in $\mathbb{Z}/(p)[x]$ is easy if you allow randomization, thanks to the work of Zieler, Berlekamp, and many others, dating back to the 1960's.
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- $\mathbb{Z}/(p^k)[x]$ is not a unique factorization domain, so we have to be more careful when counting in $\mathbb{Z}/(p^k)[x]$.
  - Example: $(x+3)^2 = x(x+6) \mod 9$.

Background to Algorithm

Consider the Taylor expansion of a polynomial $f \in \mathbb{Z}[x]$ of degree d, where $\zeta \in \mathbb{Z}$ is a root of the mod p reduction of f and $\varepsilon \in \{0, \ldots, p^k - 1\}$:

$$f(\zeta + p\varepsilon) = f(\zeta) + f'(\zeta)p\varepsilon + \cdots + \frac{1}{(k-1)!} f^{(k-1)}(\zeta)(p\varepsilon)^{k-1} \bmod p^k.$$

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Lemma (Hensel's Lemma)

If $f \in \mathbb{Z}[x]$ is a polynomial with integer coefficients, p is prime, and $\zeta_J \in \{0, \ldots, p^{J-1} - 1\}$ is a root of $f \pmod{p^J}$ and $f'(\zeta_J) \neq 0 \pmod{p}$, then there is a unique $\zeta \in \{0, \ldots, p^{J+1} - 1\}$ with $f(\zeta) = 0 \pmod{p^{J+1}}$ and $\zeta = \zeta_J \pmod{p^J}$.

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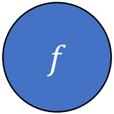
- When $s \in \{2, \ldots, k-1\}$, we can reapply the algorithm to an instance of counting roots for the polynomial $f_\zeta(\varepsilon) = \frac{f(\zeta)}{p^s} + \frac{f'(\zeta)}{p^{s-1}}\varepsilon + \cdots + \frac{f^(k-1)(\zeta)}{(k-1)!p^{s-(k-1)}}\varepsilon^{k-1}$ in $\mathbb{Z}/(p^{k-s})$.

Illustration of Complexity Bound



With $p = 3, k = 7$:

$$f(x) = x^{10} - 10x + 738$$
$$f(x) = x(x+2)^9 \mod 3$$

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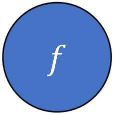

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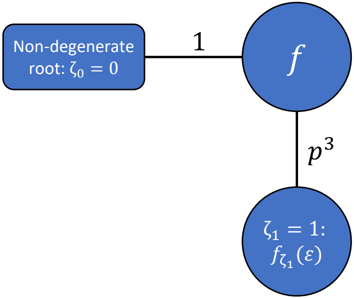

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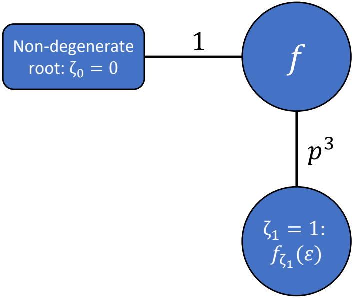

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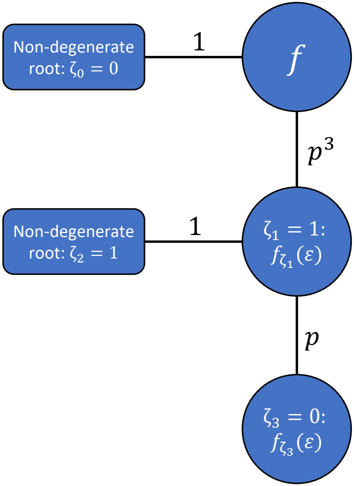

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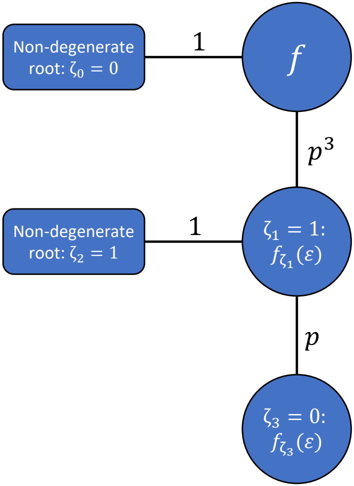

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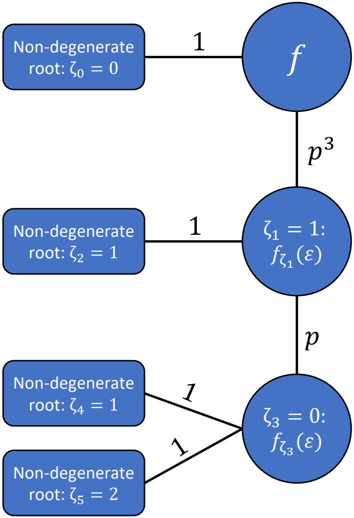

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Ideas Behind Proof of Complexity Bound

- We use Kedlaya-Umans fast $\mathbb{Z}/(p)[x]$ factoring algorithm, which takes time $d^{1.5+o(1)}(\log p)^{1+o(1)} + d^{1+o(1)}(\log p)^{2+o(1)}$ for a degree d polynomial.

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- To simplify, the complexity is less than or equal to (the number of nodes in the recursion tree)(the complexity of factoring over $(\mathbb{Z}/(p))[x]$).
- The depth and branching of our recurrence tree is strongly limited by the value of k.
- Optimizing parameters, the worst case is when $d \approx e \approx 2.71828$ and the depth is $\frac{k}{e}$.

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Example: The polynomial $f(x) = (x-2)^7(x-1)^3$ with $p = 17, k = 7$ has $24,221,090$ roots (this is greater than $\lfloor \frac{d}{k} \rfloor p^{k-1} = 24,137,569$). Computing this number of roots took .004 seconds, but storing and listing them all would take much longer.

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(Counting the number of roots using brute force took 39 minutes).

Comparing Methods

- The randomized algorithm counts the number of roots in time $d^{1.5+o(1)}(\log p)^{2+o(1)}(1.12)^k$.
- In comparison, brute force counting takes time $\approx p^k$.
- We expect the randomized algorithm to be faster even for p as small as 2 (1.12^k vs. 2^k).

Data for $p = 2$

The table below shows the average difference in computation time for the number of roots of 100 random polynomials of degree less than or equal to 100 in $\mathbb{Z}/(2^k)$ for the given k, between brute force and the randomized algorithm (negative implies brute force was faster):

k	8	9	10	11	15
Avg Diff (in seconds)	-0.0011	-0.00029	0.0028	0.01701	0.32499

Timing Data

- We expect the randomized algorithm to take the longest when a polynomial has many degenerate roots because a polynomial of this type will require many recursive calls.
- Polynomials with many degenerate roots do take longer than a random polynomial, but overall the randomized algorithm still outperforms other methods.

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- Polynomials with many degenerate roots do take longer than a random polynomial, but overall the randomized algorithm still outperforms other methods.
- Example: Counting roots of $f(x) = (x-1)(x-2)^2 \cdots (x-10)^{10}$ in $\mathbb{Z}/(31^{10})$ took 6.4 seconds using the randomized algorithm.
- For comparison, a random polynomial of the same degree (55) took 1 millisecond with the randomized algorithm, and counting roots using brute force for just 31^6 took 2.7 hours.

Example with Large Number of Roots

The number of roots of a polynomial in $\mathbb{Z}/(p^k)$ can be very large, especially for polynomials with many degenerate roots mod p.

Example: $(x-2)^{50}$ has 5132842958629010337866366828195 (31 digit number) roots in $\mathbb{Z}/(25^{29})$.

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But we do have an upper bound on the number of roots based on the sizes of k, p and d.

Steps Toward a Bound on the Number of Roots

Lemma

If a root ζ of the mod p reduction of f has multiplicity j, then $s_\zeta \leq j$, where s_ζ is the greatest integer such that p^{s_ζ} divides each of $f(\zeta), \ldots, \frac{f^{(k-1)}(\zeta)}{(k-1)!}p^{k-1}\varepsilon^{k-1}$.

Proof: If ζ has multiplicity j, then $f(\zeta) = \cdots = f^{j-1}(\zeta) = 0 \pmod{p}$, but $f^{(j)}(\zeta) \neq 0 \pmod{p}$. So $\frac{f^j(\zeta)}{j!}p^j$ is divisible by p^j but not p^{j+1} and therefore $s_\zeta \leq j$.

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This is important because it tells us that we can have at most $\lfloor \frac{d}{s} \rfloor$ roots of f mod p for each s.

Bound on the Number of Roots

Theorem

Let p be a prime, $f \in \mathbb{Z}[x]$ a polynomial of degree d, and $k \in \mathbb{N}$ such that $d \geq k \geq 2$. Then the number of roots of f in $\mathbb{Z}/(p^k)$ is less than or equal to $\min\{d, p\}p^{k-1}$.

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We know there are polynomials with more than $\min\{\lfloor \frac{d}{k} \rfloor, p\}p^{k-1}$ roots in $\mathbb{Z}/(p^k)$, so our bound is within a factor of k of optimality.

Root Bound Examples

Polynomials with $d \geq p$ having p^k roots in $\mathbb{Z}/(p^k)$:

1. $f(x) = (x^p - x)^k$ is a polynomial of degree pk with p^k roots in $\mathbb{Z}/(p^k)$.

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1. $f(x) = (x^p - x)^k$ is a polynomial of degree pk with p^k roots in $\mathbb{Z}/(p^k)$.
2. $g(x) = (x^{p^k - p^{k-1}} - 1)x^k$ has degree $p^k - p^{k-1} + k$ and also vanishes on all of $\mathbb{Z}/(p^k)$.

Sharp Bound for $k = 2$

When $k = 2$, there are a maximum of $\min\{\lfloor \frac{d}{2} \rfloor, p\}p + (d \bmod k)$ roots in $\mathbb{Z}/(p^2)$.

This upper bound is sharp. For example, the with $p = 5$ the degree 3 polynomial $(x-1)^2x$ has $\lfloor \frac{3}{2} \rfloor \cdot 5 + (3 \bmod 2) = 6$ roots in $\mathbb{Z}/(p^2)$.

Conclusions

- We have a Las Vegas randomized algorithm for counting the number of roots of a degree d polynomial $f \in \mathbb{Z}[x]$ in $\mathbb{Z}/(p^k)$.
- The complexity of the randomized algorithm is given by $\mathcal{O}(1.12^k)$.
- We see time improvements for computations using the randomized algorithm over brute-force counting even for $p = 2$.
- An upper bound on the number of roots is $\min\{d, p\}p^{k-1}$, and a sharp upper bound for $k = 2$ is given by $\min\{\lfloor \frac{d}{2} \rfloor, p\}p + (d \bmod k)$.