Parallel SSEP and a Casimir Element of $\mathfrak{so}_{2n}$

Mark Landry, Andrew Park

Texas A&M University

July 20, 2020

Goals

1. Review relevant concepts from Markov processes and Lie algebras.
2. Show the process used to start from a Casimir element of $\mathfrak{so}_{2n}$ and arrive at a generator matrix.
3. Describe expected properties of the Markov process given the generator matrix.

Markov Processes and Generator Matrices

A **Markov process** is a continuous-time physical process with a discrete number of states where states jump to other states at random times and the probabilities depend only on the present state, not past states.

A **generator matrix** encodes the jump rates between states. It has the properties:

- Each row sums to 0.
- All diagonal entries are non-positive.
- All off-diagonal entries are non-negative.

Proposition

Let Q be a generator matrix, and let q_{xy} be the (x, y) entry of Q. Let T_x be the holding time at state x. If $q_{xx} \neq 0$, then $P(X_{T_x} = y | X_0 = x) = \frac{q_{xy}}{-q_{xx}}$.

A Simple Generator Matrix Example

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

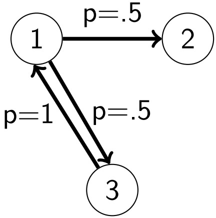


Figure: Sample Markov Process

Example: Symmetric Simple Exclusion Process (SSEP)

- Introduced by Frank Spitzer (1970)
- 2-site generator matrix derived from a Casimir element of $\mathfrak{sl}_2$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

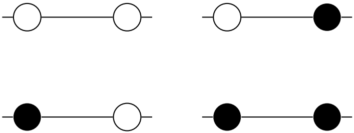
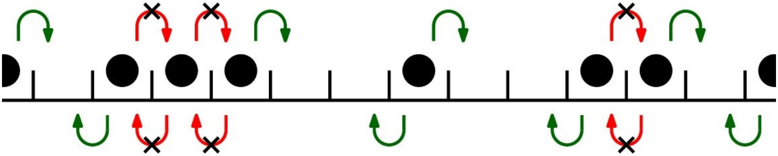


Figure: SSEP Configurations

- Can be expanded to N sites



$\mathfrak{so}_{2n}$ as a Lie Algebra

$\mathfrak{so}_{2n}(\mathbb{C})$ is the Lie algebra of matrices of the form:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A, B, C, D \in \mathbb{C}^{n\times n}, A = -D^T, B^T = -B, C^T = -C \right\}$$

Let $E_{i,j}$ be the $2n \times 2n$ matrix with a 1 in the (i,j) entry and 0 elsewhere.

- $H_i = E_{i,i} - E_{n+i,n+i}$
- $X_{ij} = E_{i,j} - E_{n+j,n+i}$
- $Y_{ij} = E_{i,n+j} - E_{j,n+i}$
- $Z_{ij} = E_{n+i,j} - E_{n+j,i}$

A Cartan-Weyl Basis of $\mathfrak{so}_{2n}$ consists of H_i for all $i \leq n$ and $X_{ij}, X_{ji}, Y_{ij}, Z_{ij}$ for all $i < j \leq n$.

Procedure to Obtain $\rho(\Omega)$

- Dual Basis (Fulfills condition with respect to Killing form)
  - Killing form $B(F,S) = (2n-2)Tr(FS)$ is 1 if S is dual basis counterpart of F, 0 for S is in dual basis but not counterpart
  - $H_i \to \frac{1}{4n-4} H_i$
  - $X_{ij} \to \frac{1}{4n-4} X_{ji}$; $X_{ji} \to \frac{1}{4n-4} X_{ij}$
  - $Y_{ij} \to -\frac{1}{4n-4} Z_{ij}$; $Z_{ij} \to -\frac{1}{4n-4} Y_{ij}$
- Casimir Element $\Omega = \sum_i A_i A^i$, where A_i is from basis and A^i is counterpart in dual basis.
- The ρ Representation
  - For $A \in \mathfrak{so}_{2n}$, $\rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(A) = \rho_{\mathbb{C}^{2n}}(A) \otimes Id_{2n} + Id_{2n} \otimes \rho_{\mathbb{C}^{2n}}(A)$
  - Example: $\rho_{\mathbb{C}^{2n}}(X_{12})$ is a $2n \times 2n$ matrix, $\rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(X_{12})$ is a $4n^2 \times 4n^2$ matrix.
- Compute $\rho(\Omega) = \sum_i \rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(A_i) \rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(A^i)$

Block Form of $\rho(\Omega)$

Lemma (Landry-Park)

The representation of the Casimir element $\rho(\Omega)$, a $4n^2 \times 4n^2$ matrix, can be written as a $2n \times 2n$ matrix of blocks each of size $2n \times 2n$ as:

$$\rho(\Omega) = \frac{1}{2n-2}\left(\begin{array}{ccccc|ccccc} D_1 & X_{21} & X_{31} & \cdots & X_{n,1} & 0 & -Z_{12} & -Z_{13} & \cdots & -Z_{1,n} \\ X_{12} & D_2 & X_{32} & \cdots & X_{n,2} & Z_{12} & 0 & -Z_{23} & \cdots & -Z_{2,n} \\ X_{13} & X_{23} & D_3 & \cdots & X_{n,3} & Z_{13} & Z_{23} & 0 & \cdots & -Z_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1,n} & X_{2,n} & X_{3,n} & \cdots & D_n & Z_{1,n} & Z_{2,n} & Z_{3,n} & \cdots & 0 \\ \hline 0 & -Y_{12} & -Y_{13} & \cdots & -Y_{1,n} & D_{n+1} & -X_{12} & -X_{13} & \cdots & -X_{1,n} \\ Y_{12} & 0 & -Y_{23} & \cdots & -Y_{2,n} & -X_{21} & D_{n+2} & -X_{23} & \cdots & -X_{2,n} \\ Y_{13} & Y_{23} & 0 & \cdots & -Y_{3,n} & -X_{31} & -X_{32} & D_{n+3} & \cdots & -X_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Y_{1,n} & Y_{2,n} & Y_{3,n} & \cdots & 0 & -X_{n,1} & -X_{n,2} & -X_{n,3} & \cdots & D_{2n} \end{array}\right)$$

where $D_i = (2n-1)I + H_i$ and $D_{n+i} = (2n-1)I - H_i$ for $i \leq n$.

From $\rho(\Omega)$ to G_n

$\rho(\Omega)$ is not yet a generator matrix, as its rows do not yet sum to 0. We need to use a correction term C such that $\rho(\Omega) - C$ is a generator matrix.

Procedure

- Let C be the diagonal matrix with entries c_i such that $c_i = \sum_j \rho(\Omega)_{ij}$.
- Negate rows of $\rho(\Omega) - C$ to force all diagonal elements to be non-positive. Call this matrix G_n.

Note: Past research uses $C = kI$ for some $k \in \mathbb{C}$, and then conjugates with a diagonal matrix to arrive at a generator matrix. In the case of $\mathfrak{so}_{2n}$, this technique fails to arrive at a generator matrix with finite entries if $n > 2$.

Properties of G_n

Lemma (Landry-Park)

The matrix G_n is the generator of a Markov process.

Lemma (Landry-Park)

All non-zero off-diagonal entries of G_n are equal.

First Properties of Markov Process:

- G_n is $4n^2 \times 4n^2$, which implies a Markov process with $4n^2$ states.
- From past research on Lie algebras, Casimir representations describe a particle system with two sites.

Absorbing States

An **absorbing state** of a Markov process is a state that "absorbs," i.e. if the process lands there, it will never jump to another state. This is represented in a generator matrix by a row of 0's.

Lemma (Landry-Park)

The Markov process with generator G_n has $2n$ absorbing states.

Figure: 2 Absorbing States in SSEP

Maximal Choice Rows

Definition (Landry-Park)

A **maximal choice row** is a row in which no other rows have a greater number of nonzero off-diagonal elements. The set of all maximal choice rows is called the **maximal choice set**.

Lemma (Landry-Park)

The generator G_n has $2n$ maximal choice rows, each of which have $2n - 2$ non-zero off-diagonal entries.

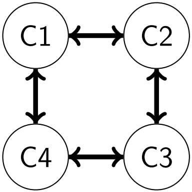


Figure: 4 Maximum Choice States

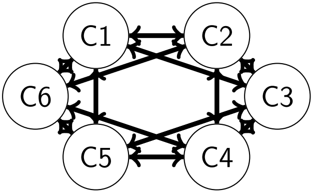


Figure: 6 Maximum Choice States

Pairwise Rows and Pairwise States

Definition (Landry-Park)

A **pairwise row** is a row with exactly one nonzero off-diagonal entry. If pairwise row r has its nonzero off-diagonal entry at column s, and s is a pairwise row with nonzero off-diagonal entry at column r, rows r and s are called **pairwise states**.

Lemma (Landry-Park)

Any row in G_n is either an absorbing state, or a maximal choice row, or a pairwise state.

Figure: 2 Pairwise States from SSEP

Summary

Based on the properties we observe in G_n, a generator matrix derived from a Casimir element in $\mathfrak{so}_{2n}$, we expect the following properties in the corresponding Markov process:

- $4n^2$ states total.
- $2n$ absorbing states.
- $2n$ maximum choice states that can each jump to $2n - 2$ other maximum choice states.
- The remaining states split up into pairs that jump back and forth.

Parallel SSEP

Definition (Landry-Park)

A **Parallel SSEP with N sites** is a system that has two separate 1-dimensional SSEPs with $N \geq 2$ sites each. Each site can either be empty or have a particle on it, and while particles can interact with neighboring sites on the same lattice, they cannot jump to the other lattice. A Parallel SSEP with 2 sites will be referred to as a **Basic Parallel SSEP**.

Type-1 Parallel SSEP is the simplest case of a Parallel SSEP; it is a Parallel SSEP with only 1 type of particle.

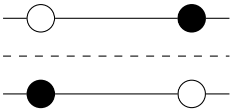


Figure: A possible state of a Type-1 Basic Parallel SSEP

Type-2 Basic Parallel SSEP

Consider a system similar to the Type-1 Basic Parallel SSEP where we introduce a second type of particle, 1/2 the mass of the first particle, to the upper lattice. This is the **Type-2 Basic Parallel SSEP** and has the following properties:

1. Heavier particles can not move as long as a lighter particle in the system can move.
2. If the two lattices are equal by mass, then the upper lattice allows fusion/fission which provides energy, possibly granting the lower lattice a free concurrent move.
3. Lighter particles have the "upper-class" property seen in other SSEP variants.

Type-2 Basic Parallel SSEP

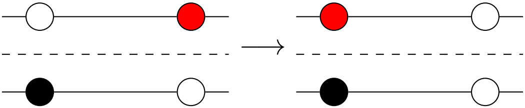


Figure: The black particle is unable to move

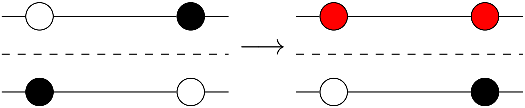


Figure: The two lattices are balanced; the top particle can undergo fission

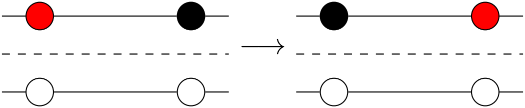


Figure: The red particle will switch places with the black particle

Type-m Basic Parallel SSEP

Consider a Basic Parallel SSEP where the lower lattice allows particles of mass 1, and the upper lattice allows particles of mass $\omega \in \{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\}$. We define the following properties:

1. **Mass Order Property**: A particle can only move if no lighter particles can move.
2. **Balance Property**: A set of balanced states exists in which the two lattices each have mass 1 and particles are able to undergo fusion and fission, defined respectively as donating mass to or taking mass from a neighboring site. These mass-preserving processes allow a concurrent move in the lower lattice.
3. **Class Property**: A particle in a non-balanced state is able to switch places with neighboring particles of higher mass.

Type-m Basic Parallel SSEP

Definition (Landry-Park)

A **Type-m Basic Parallel SSEP** *is a Basic Parallel SSEP where the lower lattice allows particles of mass* 1*, the upper lattice allows particles of mass* $\omega \in \{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\}$*, and the following properties hold: Mass Order Property, Balance Property, and Class Property.*

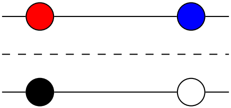


Figure: A possible state of a Type-3 Basic Parallel SSEP with mass values: $\frac{1}{3}$, $\frac{2}{3}$, 1

The Connection

We return to the generator G_n of a Markov process, and its connection to our newly defined particle system:

Theorem (Landry-Park)

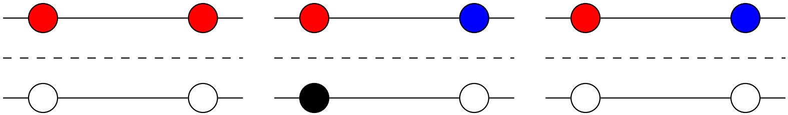
Let $m = n - 1$. The generator matrix of the Type-m Basic Parallel SSEP is exactly G_n.

Evidence

Lemma (Landry-Park)

A Type-m Basic Parallel SSEP has $4(m+1)^2$ *states made up of:*

- $2(m+1)$ *absorbing states*
- $2(m+1)$ *maximum-choice states*
- $4(m+1)^2 - 4(m+1)$ *pairwise states*



(a) Absorbing State (b) Max-choice State (c) Pairwise State

Figure: Different states of Type-3 Basic Parallel SSEP with mass values: $\frac{1}{3}$,$\frac{2}{3}$,1

Expansion of the Particle System

The following formula takes L, the generator matrix for the SSEP with 2 sites, and expands it to L_N, the generator matrix for the SSEP with N sites:

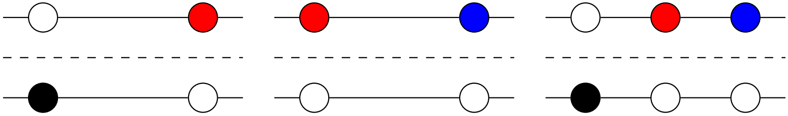
$$L_N = \sum_{j=0}^{N-2} \underbrace{I \otimes \cdots \otimes I}_{j} \otimes L \otimes \underbrace{I \otimes \cdots \otimes I}_{N-j-2}$$

This formula also holds for the generator of a Parallel SSEP! Note that each lattice is expanded to N sites, so L_N gives $2N$ sites in total for a Parallel SSEP.

Subsystems

Definition (Landry-Park)

A **subsystem** of a Parallel SSEP with N sites is a subset of the system such that it is a Basic Parallel SSEP. A Parallel SSEP with N sites is made up of $N-1$ overlapping subsystems which define the local properties of the entire system.



(a) Subsystem 1 (b) Subsystem 2 (c) Entire System

Figure: How subsystems make up the entire system

Expanded Type-m Parallel SSEP

Definition (Landry-Park)

A **Type-m Parallel SSEP with N sites** is a Parallel SSEP with N sites such that every subsystem is a Type-m Basic Parallel SSEP.

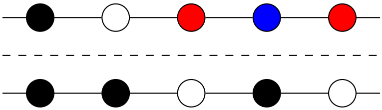


Figure: A possible state of a Type-3 Parallel SSEP with 5 sites

Remark

The total number of choices available to a state of a Type-m Parallel SSEP with N sites is just the sum of the number of choices of its subsystems.

The Final Result

Theorem (Landry-Park)

Let $L = G_n$, the generator of a Type-m Basic Parallel SSEP, and let:

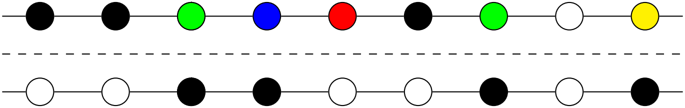
$$L_N = \sum_{i=1}^{N-1} \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes L \otimes \underbrace{I \otimes \cdots \otimes I}_{N-1-i},$$

L_N is the generator of a Type-m Parallel SSEP with N sites.

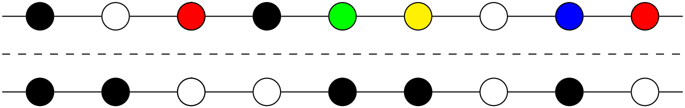
Remark

The i^{th} term of the summation corresponds directly to the i^{th} subsystem!

Thank you!



Thank you to:
Texas A&M University
Prof. Jeffrey Kuan
TA's Ola Sobieska and Zhengye Zhou
National Science Foundation (DMS-1757872)



References

Gioia Carinci, Cristian Giardinà, Frank Redig, and Tomohiro Sasamoto.
A generalized asymmetric exclusion process with $U_q(\mathfrak{sl}_2)$ stochastic duality.
Probab. Theory Related Fields, 166(3-4):887–933, 2016.

Jeffrey Kuan.
Stochastic duality of ASEP with two particle types via symmetry of quantum groups of rank two.
J. Phys. A, 49(11):115002, 29, 2016.

Frank Spitzer.
Interaction of markov processes.
Advances in Mathematics, 5(2):246, 45, 1970.