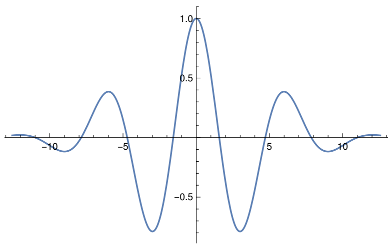

Figure 1: Graph of damped cosine wave

Solutions for the Texas A&M Freshman-Sophomore Contest 2017

First year student version

There are two pages, six problems. The first three problems are common to both versions.

1. Let $f(x) = \cos(x)e^{-x^2/(4\pi^2)}$.

   (a) Sketch the graph of $f(x)$ over the interval $[-4\pi, 4\pi]$.

   (b) Find the derivative of $f(x)$ at $x = \pi$ and simplify fully. The product rule and the chain rule come into play because f is the product of $\cos x$ and $e^{-x^2/4\pi^2}$. The derivative works out to $(-x\cos x/(2\pi^2) - \sin x)e^{-x^2/4\pi^2}$. Setting $x = \pi$ zeroes out the sine term and the answer is $\frac{1}{2\pi}e^{-1/4}$.

2. The identity $\cos(2t) = 2\cos^2(t) - 1$ has some curious consequences.

   (a) Express $\cos(4t)$ in terms of $\cos t$. It's $8\cos^4(t) - 8\cos^2 t + 1$.

   (b) Sketch the graph of $y = x^4 - x^2 + \frac{1}{8}$ on the interval $-1 \leq x \leq 1$, and find the minimum value of y and where it occurs. The minimum value is $-1/8$ because of the first two parts, which imply that this polynomial is $(1/8)\cos(4\cos^{-1} x)$. The minimum value occurs at $x = \pm 1/\sqrt{2}$ because the derivative is $2x(2x^2 - 1)$ which is zero at those places and at zero. But at 0, the original polynomial evaluates to positive. Because of the tie-in with the cosine function, the graph runs back and forth between $-1/8$ and $1/8$; the maximum occurs at 0 and at $/pm1$, while the minimum occurs at $\pm 1/\sqrt{2}$. Polynomials that agree on $[-1, 1]$ with $\cos(n\cos^{-1} x)$ are called *Chebyshev* polynomials and have all sorts of interesting properties, not just the one that defines them.

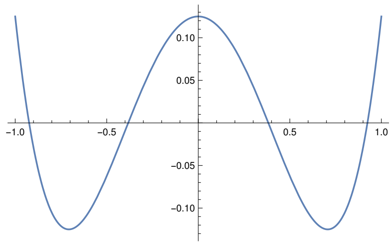

Figure 2: Graph of Cheybshev-type polynomial

3. Take as given the power series expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Find in closed form

$$A = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(2k-1)!}{(k-1)!2^{2k-1}}.$$

The $(2k-1)!$ in the numerator cancels all of the $(2k)!$ in the denominator except for its final factor $2k$. Putting the 2 here with the 2^{2k-1} gives that

$$A = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{-2k}}{k!} = e^{-1/4},$$

this last by the series expansion for e^z specialized to the case $z = -1/4$.

4. Let $g(x) = x + x^2/2^2 + x^4/4^4 + x^8/8^8 + \cdots$.

   (a) Determine with proof the radius of convergence of the series defining $g(x)$. The radius is infinity. That is, the series converges for all values of x. There are several ways to prove this. The ratio of the $k+1$th nonzero term to the kth nonzero term is

$$r_k = \frac{x^{2^{k+1}}}{2^{(k+1)2^{k+1}}} \frac{2^{k2^k}}{x^{2^k}} = \frac{x^{2^k}}{2^{(k+2)2^k}} = \left(\frac{x}{2^{j+2}}\right)^{2^k}.$$

   The last expression here is the 2^k power of an expression which is going to zero for any fixed x, as k tends to infinity. So the limit of the ratio is zero, whatever the value of x, and thus by the ratio test for ordinary series, this series converges for all x and the radius of convergence is infinite.

   The root test says, for the case at hand, that if the coefficient of x^n is a_n and $|a_n|^{1/n}$ goes to 0, then the radius of convergence is infinite.

Here, when n is not a power of 2, the coefficient is exactly 0 which is more than sufficient. If $n = 2^k$, then the coefficient is

$$a_{2^k} = (2^k)^{-2^k} = 2^{k2^k}.$$

The 2^kth root of this is 2^{-k}. In other words, for the interesting values of n, $a_n^{1/n} = 1/n$. That of course goes to zero.

Another proof would be that the coefficients are either zero, or when they are not, they have the form n^{-n} which is less than or equal to $1/n!$ because $n! = 1 \cdot 2 \cdots n \leq n \cdot n \cdots n$. Since the series for e^x converges for all x, and since the coefficients are all positive and smaller in the case at hand, this series too converges everywhere.

(b) Find, to six decimal places accuracy, $g'(1)$. The derivative of a power series can be taken term by term. Here, the derivative is $1 + 2x/4 + 4x^3/256 + 8x^7/(8^8) + \cdots$ and the next term is just $16^{-15} < 10^{-10}$ and all subsequent terms are less than half the one that came before so the total of what's not included in the initial arithmetic is less than $2 \cdot 10^{-10}$. Now

$$1 + 0.5 + 0.015625 + \text{too little to matter} = 1.515625.$$

(c) Find the integer nearest $g(6)$. The first few terms are 6, 9, and $81/16 = 5 + 1/16$. The rest are collectively too small to matter. The first one not part of this arithmetic is $(3/4)^4$ which is about $1/10$, and the ones that follow are each less than half the previous so in total they add less than $1/5$. Thus the nearest integer is 20.

(d) Prove that there are infinitely many positive integers N so that $g(N) > 2^{N/2}$. If N is a power of 2, then the expression for $g(N)$ includes the term $N^{N/2}/(N/2)^{N/2}$ as the term right before the term $N^N/N^N = 1$. But $N^{N/2}/((N/2)^{N/2}) = 2^{N/2}$. The rest of the series makes the series total greater than $2^{N/2}$.

(e) Prove that there is a constant C so that $G(N) < 2^{CN}$ for all positive N. As noted earlier, $g(x) < e^x$. Thus $g(x) < 2^{x/\ln(2)}$. Take $C = 1/\ln(2)$.

5. Graph the curve $y = (1-x)/\sqrt{1-x^2}$ on the interval $[0, 1)$ and find the volume of the solid enclosed by rotating that curve about the x axis. The function can be simplified to $\sqrt{(1-x)/(1+x)}$ which makes it easier to see what is going on. When $x = 0$, the function is at 1, and it decreases until when $x = 1$, it's at 0.

   The volume involved is $\int_0^1 \pi(1-x)/(1+x)\,dx$ because the disk method involves πr^2 and that squares the square root in the formula. Now with the substitution $u = 1 + x$, the numerator is $2 - u$ so the volume is $\pi \int_1^2 (2-u)/u\,du = \pi(2\ln(2) - 1)$.

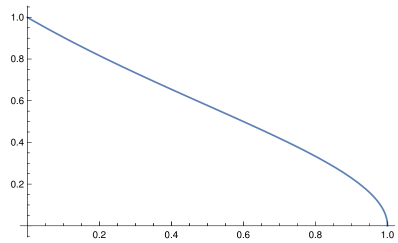

Figure 3: Graph of curve for problem 5

6. Define $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Thus, for instance, $\zeta(2) = \sum_{n=1}^{\infty} n^{-2}$.

(a) For which real numbers s does the series defining $\zeta(s)$ converge? It converges for $s > 1$. Compare the series to the improper integral in the next part of the problem. If you cut that integral into pieces of length 1, each piece comes to a value comparable to the corresponding term in the series.

(b) Let $f(s) = \int_1^{\infty} x^{-s}\, dx$. Find a closed-form expression for $f(s)$ when the improper integral converges. The antiderivative of x^a is $(1/(a+1))x^{a+1}$ except when $a = -1$. Here, $a = -s$ so that rule gives $(1/(1-s))x^{1-s}$. Evaluated at 1 and infinity that works out to $1/(s-1)$, which is the answer.

(c) Prove that this chain of statements is true for $n \geq 1$ and $s > 0$:

$$n^{-s} - \int_{x=n}^{n+1} x^{-s}\, dx < n^{-s} - (n+1)^{-s} = s \int_{x=n}^{n+1} x^{-s-1}\, dx < sn^{-s-1}.$$

The first inequality holds because on the interval $(n, n+1)$, $x < n+1$ so $x^{-s} > (n+1)^{-s}$ so subtracting x^{-s} leaves less than subtracting merely $(n+1)^{-s}$. The second claim is true because of the antiderivative rule noted already, multiplied by s. The final inequality is true because on the interval $(n, n+1)$, $x^{-s-1} < n^{-s-1}$ and the integral of that over $[n, n+1]$ is just n^{-s-1}.