TAMU Freshman-Sophomore Contest, 2016

Second-year students' version

While the name of the contest is traditional, the actual eligibility rules are that first year students take the freshman contest, and second year students take the sophomore contest. That way, students who have accumulated enough credit hours in their first or second year to have standing as sophomores, or juniors, are not promoted out of eligibility.

The first page contains problems built around Calculus I and II for both freshmen and sophomores. The second pages are pitched to content unique to Calculus III and/or Differential Equations, in the case of the second-year version.

In all cases, solutions should be written out and should include reasoning behind the steps when reasons beyond routine calculation are involved. No tables, calculators, or computers, and no devices for communication with the outside world, are allowed. You're on your own.

1. Find $\int_0^{\pi/2} \cos x \cos 2x \, dx$. The identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$, applied to the cases $a = 2x, b = x$ and $a = 2x$, $b = -x$ yields $\cos 2x \cos x = \frac{1}{2}(\cos 3x + \cos x)$. Integrating this from 0 to $\pi/2$ gives an answer of $\frac{1}{2}(\sin 3x/3 + \sin x|_0^{\pi/2} = \frac{1}{3}$. The answer is 1/3.

   For another solution, the same identity for $\cos(a+b)$ is applied instead to the case $a = b = x$, so that $\cos x \cos 2x = (\cos x)(\cos^2 x - \sin^2 x)$. But now

   $$\int_{x=0}^{\pi/2} (\cos x)(\cos^2 x - \sin^2 x) \, dx = \int_{x=0}^{\pi/2} (1 - 2\sin^2 x) \cos x \, dx$$
   $$= \int_{u=0}^{1} (1 - u^2) \, du = 1 - \frac{2}{3} = \frac{1}{3}$$

   by way of the substitution $u = \sin x$, $du = \cos x \, dx$. Which is the better solution? From one perspective, the second solution is best. It's shorter and easier to understand. From another perspective, the first solution is best because it can be adapted to a wider variety of similar cases, such as $\cos(2x)\cos(3x)$.

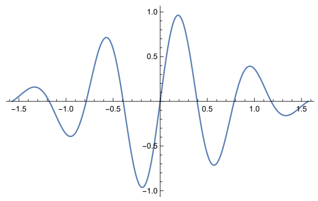
2. Let $f(x) = \frac{x}{1+x^2}$. Let $g(x)$ be the 19th derivative of $f(x)$. Find $\frac{g(0)}{20!}$. Probably the best way to work this is to first get the series expansion of $f(x)$ about 0. The series $1/(1-z) = 1 + z + z^2 + \cdots$, with $z = -x^2$, gives $\frac{x}{1+x^2} = x - x^3 + x^5 - x^7 \cdots + x^{17} - x^{19} + \cdots$. Taking the 19th derivative term by term, as we may do inside the radius of convergence, which is 1 here, gives

   $$f^{(19)}(x) = -19! + \frac{21!}{2!}x^2 - \frac{23!}{4!}x^4 - \cdots$$

and setting $x = 0$ and dividing by 20!, the answer is $-1/20$.

3. Let $u(x) = \sin(8x)e^{-x^2}$.

   (a) Graph $u(x)$ on the interval $-\pi/2 \le x \le \pi/2$.



   (b) Given that $\int_0^\infty t^k e^{-t}\,dt = k!$ for all nonnegative integers k, prove that

$$\int_0^\infty u(x)\,dx = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{k!8^{2k+1}}{(2k+1)!}.$$

Here we need to break up $\sin(8x)$ into its power series, but not break up e^{-x^2}. A truly careful proof must address matters of convergence, and that can be done here by cutting off the series at some point N and then proving that the rest of the series is at any rate bounded between what one would get with all the terms positive from then on, and with all of them negative, and that both bounds tend to zero. For all x, $\sin(8x) = \sum_{k=0}^{\infty}$. For any positive integer N, this can be split as $\sin(8x) = (\sum_{k=0}^{N} + \sum_{k=N+1}^{\infty}(-1)^k(8x)^{2k+1}/(2k+1)!$. Let the first sum be $S_N(x)$ and the second sum be $R_N(x)$, as in sum and remainder. We have $|R_n(x)| \le \sum_{k=N+1}^{\infty}(8x)^{2k+1}/(2k+1)!$. Now let's get down to the mechanics.

$$\int_0^\infty S_N(x)\,dx = \int_0^\infty \sum_{k=0}^{N}(-1)^k 8^{2k+1}x^{2k+1}e^{-x^2}/(2k+1)!\,dx$$

$$= \sum_{k=0}^{N}\left((-1)^k 8^{2k+1}/(2k+1)!\right)\int_{x=0}^{\infty} x^{2k+1}e^{-x^2}\,dx$$

$$= \frac{1}{2}\sum_{k=0}^{N}\left((-1)^k 8^{2k+1}/(2k+1)!\right)\int_{u=0}^{\infty} u^k e^{-u}\,du$$

$$= \frac{1}{2}\sum_{k=0}^{N}(-1)^k 8^{2k+1}\frac{k!}{(2k+1)!}.$$

Also, $\int_0^\infty R_N(x)\,dx < \sum_{k=N+1}^\infty 8^{2k+1}\frac{k!}{(2k+1)!}$, because with everything positive, we can switch summation and integration with confidence. If any version converges then the others do as well, and all of them, with absolute convergence. For $N > 10$, say, each term is less than half the one before it in this sum, so the total is less than twice the first term. That is, $\int_0^\infty R_N(x)\,dx < 2 \cdot 8^{2(N+1)+1}\frac{(N+1)!}{2N+3)!}$, an expression which tends to 0 rapidly as N tends to infinity because the factorial of $2N+3$ dominates.

So for all N, $\int_0^\infty u(x)\,dx = \int_0^\infty S_N(x) + R_N(x)\,dx$. The first integral evaluates to an expression whose limit is the claimed answer, and the second integral evaluates to an expression whose limit is 0. We are done.

4. Find the average value of the distance from a point on the sphere of radius 1 about the origin to the plane $z = 0$. Average so as to give equal weight to equal areas on the sphere.

   The average distance is 1/2. To find the average, we integrate the distance and divide by the area 4π of the sphere. But–how shall we set up the integral? To properly give equal weight to equal areas, we should integrate $|z|$ with respect to surface area on the sphere.

   The element of surface area in spherical coordinates is $\rho^2 \sin\phi\, d\theta\, d\phi$. We can just integrate over the top hemisphere and then double that. Now on our sphere, $\rho = 1$, and $z = \rho\cos\phi = \cos\phi$. So our answer is

$$2\frac{1}{4\pi}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} 1^2 \sin\phi\cos\phi\, d\theta\, d\phi.$$

   This simplifies down to $\int_{\phi=0}^{\pi/2} \sin\phi\cos\phi\, d\phi = 1/2$ with the substitution $u = \sin\phi$, $du = \cos\phi\, d\phi$. The answer is 1/2.

   While we're on the topic of surface area and z, there's a nice little metaphor for a result that says that the surface area on the sphere between any two planes $z = a$ and $z = b$ is directly proportional to $b - a$: a tomato slicer cuts the tomato into slices that each have the same amount of skin.

5. Let $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. Let C be the path going counterclockwise once about the ellipse $x^2/9 + y^2/4 = 1$. Find

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

   There are two ways to go at this. One approach is to use Green's theorem. The path integral is equal to the double integral over the interior of the ellipse of $\partial N/\partial x - \partial M/\partial y$. Here, $M = y$ and $N = -x$ so that expression works out to -2. The area of an ellipse is given by a formula quite similar to the πr^2 formula for the area of a circle, with the twist that instead of a single radius r the maximum and minimum radius a and b are used. (The radius measured along the long and short axis, in other words.) Here, $a = 3$ and $b = 2$ so the area is 6π and the answer is -12π.

The other approach would be to evaluate the integral directly by getting a parametrization of the ellipse: $x = 3\cos t$, $y = 2\sin t$, with t running 0 to 2π. That gives $\int_0^{2\pi}(2\sin t\mathbf{i} - 3\cos t\mathbf{j})\cdot(-3\sin t\mathbf{i} + 2\cos t\mathbf{j})\,dt = -6\int_0^{2\pi}\sin^2 t + \cos^2 t\,dt = -12\pi$. The answer is still -12π.

6. Let $x(t)$ and $y(t)$ be functions of t satisfying initial conditions $x(0) = 6$, $y(0) = 5$, and the differential equations

$$\frac{dx}{dt} = -y^2(t)$$
$$\frac{dy}{dt} = -x^2(t).$$

   (a) Find d^2x/dt^2 in terms of $x(t)$ and $y(t)$. It's $2x^2(t)y(t)$.

   (b) There is (a unique) $T > 0$ such that $y(T) = 4$. Find $x(T)$. Consider the expression $x^3(t) - y^3(t)$. This has derivative $3x^2(t)dx/dt - 3y^2(t)dy/dt = -3x^2y^2 - 3y^2(-x^2) = 0$. So $x^3 - y^3$ isn't changing. It starts out at $216 - 125 = 91$ when $t = 0$, and when $t = T$, $y = 4$ so x is whatever it has to be to make $x^3 - 64 = 91$. That is, $x(T) = 155^{1/3}$.

   (c) Prove that on the interval $0 < t < T$, $x(t) - y(t)$ is increasing. The derivative of $x - y$ is $x^2 - y^2 = (x - y)(x + y)$. Now x and y are both decreasing since their derivative is the in both cases the negative of a squared quantity. Thus for $0 \le t \le T$, $4 \le y \le 5$, and thus $5 < 155^{1/3} \le x \le 6$. Since $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ and both $x^3 - y^3$ and $x^2 + xy + y^2$ are positive, $x - y$ is also positive. This means that $(x - y)' = (x^2 - y^2) = (x - y)(x + y)$ is a product of positive quantities and thus positive. Therefore, $x - y$ is increasing.

   (d) Prove that $T < 1/10$. What about $T < 1/25$? What about $T < 1/50$? For $0 \le t \le T$, $5 < x \le 6$, so $-36 \le y' < -25$. With a derivative always more negative than -25, it cannot take y as long as $1/25$ to drop by 1, so $T < 1/25$.

   On the other hand, with a derivative always greater than -36, y cannot decrease by 1 over a t-interval shorter than $1/36$. Therefore, $1/36 < T$. So $1/50 < T < 1/25 < 1/10$.